

Hamiltonians of Persistent Interactions

THESE

présentée à la faculté des sciences
de l'Université de Genève
pour obtenir le grade de docteur ès sciences physiques
par

JEAN-PIERRE ECKMANN

de
Berne (BE)

Thèse No 1510

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La faculté des sciences, sur le préavis de M. M. GUENIN, prof. assistant, directeur de thèse, et d'une commission composée des professeurs J.M. JAUCH et K. HEPP (de l'ETH à Zürich), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

Genève, le 5 février 1970

Le doyen: Marc VUAGNAT

Thèse No. 1510



Résumé : Nous discutons quelques problèmes mathématiques d'un modèle particulier en théorie de champ. La classe de modèles qui sera traitée a des Hamiltoniens de la forme $H = H_0 + V$

$$H_0 = \int \omega(k) b^*(k) b(k) d^s k + \int \mu(k) a^*(k) a(k) d^s k$$

$$V = \int \phi_b^+(x) \phi_b^-(x) : P(\phi_a) : (x) d^s x$$

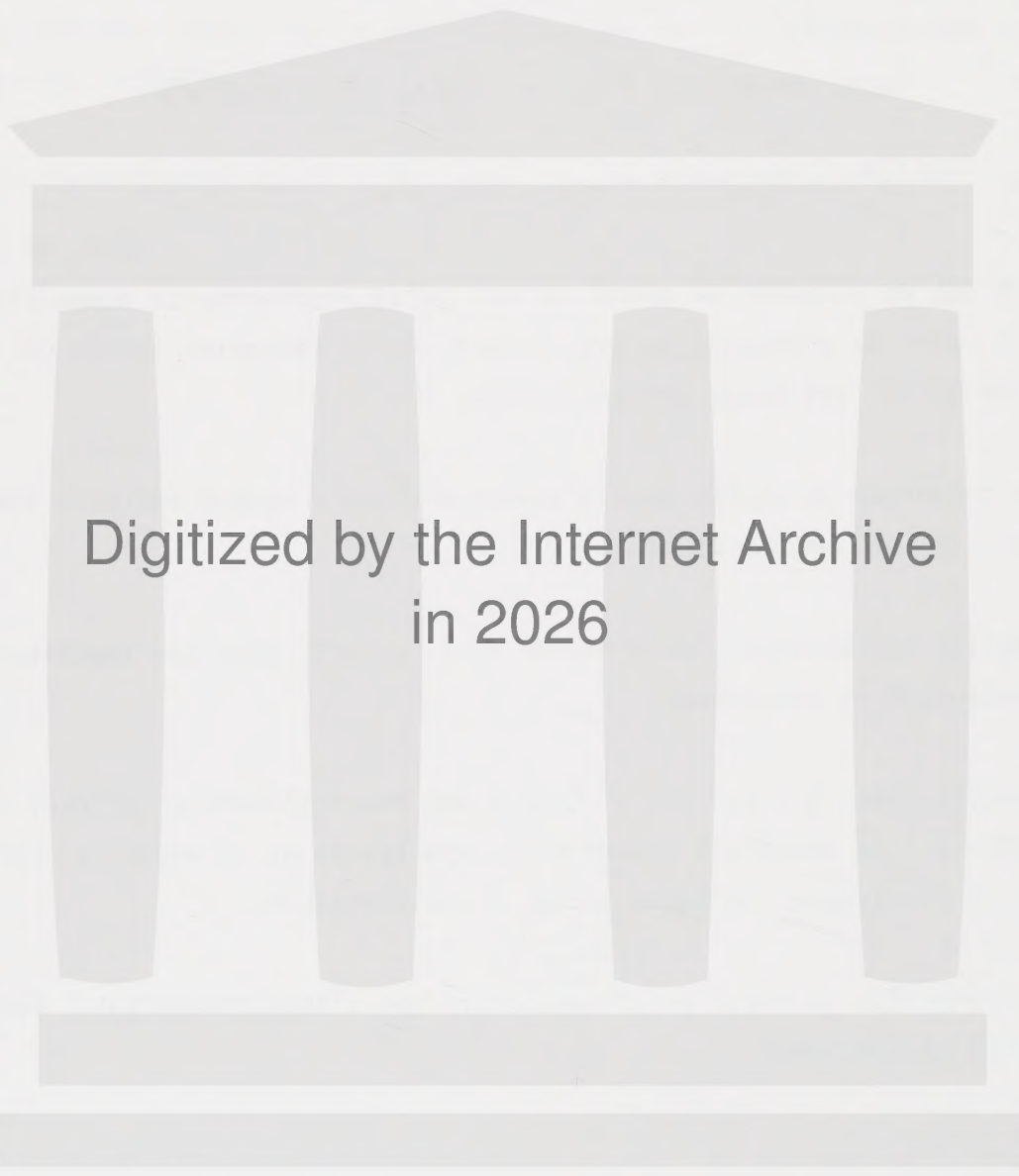
Ici P est un polynôme de degré d et $\phi_i(x)$ représente le champ libre à temps $t = 0$ pour la particule de type a ou b . On se restreint au cas où $d = 1$ ou au cas où P est borné par en-dessous.

Nous comparons ce modèle avec d'autres modèles : modèle scalaire sans recul, modèle de Lee, modèle de Nelson.

On montre que quelques Hamiltoniens avec "cutoff" dans les impulsions sont autoadjoints et semibornés.

Le modèle avec $d = 1$, $s = 3$ (qui a une renormalisation infinie) est discuté en détail : on construit un opérateur symétrique et on somme la série de Born pour la résolvante. On donne l'état à une particule.

Pour le modèle $d = 2$, $s = 1$ on montre que l'Hamiltonien total est auto-adjoint et semiborné.



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Abstract: Persistent models have Hamiltonians of the form $H = H_0 + V$, where

$$H_0 = \int \omega(k) b^*(k) b(k) d^s k + \int \mu(k) a^*(k) a(k) d^s k$$

$$V = \int \phi_b^+(x) \phi_b^-(x) : P(\phi_a) : (x) d^s x$$

and where P is a polynomial, a and b label two different boson types and ϕ_i , $i \in \{a, b\}$ denote the corresponding free fields. $\phi^+ = (\phi^-)^*$ is the creation part of the free field. We discuss the properties of such models. The cutoff Hamiltonians are shown to be selfadjoint operators for $\deg P = 1, 2, 4$ and $s = 1$. Without cutoff they are semibounded for $\deg P = 2$, $s = 1$. Most information can be obtained for the model $\deg P = 1$, $s = 3$ which has an infinite mass renormalization. For this model the Hamiltonian is shown to be selfadjoint in the limit of no cutoff by summing the Born series. A one particle state is presented.

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I. INTRODUCTION

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Quantum field models date back to the early times of quantum mechanics, but only in recent years has a completely rigorous treatment (in Fock space) of their properties been developed. This created a great demand for new mathematical tools among which the most outstanding are careful estimates of operators and their powers, and perturbation theory of operators going beyond the scope of Kato's theorem.

We shall study some of these mathematical problems as they appear in a more simple model than those models offered by relativistic quantum field theory. In fact, our model (called persistent) will not show the difficulties in connection with Haag's theorem. The technical difficulties which will arise as we discuss the persistent models are, however, more complex than those found with the Lee model, because in the persistent models one finds creation of infinitely many particles from finitely many. This is in sharp contrast to the Lee model where the particle number can change by a finite amount only.

One of the most important features which has to be established in the context of constructive quantum field theory is the existence of a Hamiltonian as a "local", selfadjoint and semibounded operator. A further stage is the discussion of the spectrum of this operator, especially with a view to the one-particle structure of the theory.

In Chapter II we shall discuss the notation and the definition for a class of models, which will be called "persistent", in view of the fact that the physical and the free vacuum are identical. Persistent interactions describe two types of bosons, called a and b and a Hamiltonian of the form

$$H = H_0 + V, \text{ where}$$

$$\begin{aligned}
H_0 &= \int \omega(k) b^*(k) b(k) d^s k + \int \mu(k) a^*(k) a(k) d^s k \\
V &= \int \phi_b^+(x) \phi_b^-(x) : P(\phi_a) : (x) d^s x
\end{aligned}$$

and where P is a polynomial of degree d and $\phi_i(x)$ is the free field at time zero for the particle type $i = a$ or b . We restrict our considerations to the case $d = 1$ or $d = \text{even}$ with P a lower semibounded polynomial $\neq 0$.

Chapter III gives some connections between persistent models and other well-known field models. (Scalar field model without recoil, Lee model, Nelson model.)

In Chapter IV we discuss the Hamiltonians which are cutoff in momentum space. We prove selfadjointness and semiboundedness of the total Hamiltonian.

Chapter V treats the case $d = 1$ without cutoff (and with infinite renormalization). We prove selfadjointness of the total Hamiltonian in $s = 3$ space dimensions.

Chapter VI contains a proof of the semiboundedness of the total Hamiltonian without space cutoff for the case $d = 2, s = 1$. By the same proof we find that the total Hamiltonian for this model is selfadjoint for small coupling constant. We give a discussion of the one-particle structure for the model treated in Chapter V.

As a conclusion one might say that the following two goals have been reached. We begin to have control over models with two types of particles without any "dominating" single particle interaction in contradistinction to the results of Jaffe, Lanford and Wightman [15]. Secondly, the convergence of the Born series is not limited to the cases where the interaction can change the particle number by a finite amount only, as is the case with the Lee model (Schrader [21], Hepp [12]). The convergence proof can be extended to the cases where the Hamiltonian increases the particle number by one each time it is applied. For this model, a one-particle state can be constructed, provided the coupling constant is small enough.

I am grateful to Profs. M. Guenin and K. Hepp for their constant encouragement and their most useful comments in the course of this work. It is a pleasure to thank Prof. W. Faris for all he taught me in innumerable discussions on operator theory.

II. DEFINITION OF PERSISTENT MODELS

II.1. Notations for Second-Quantized Operators

We begin with the construction of the Fock space $\mathcal{F}$ for one scalar field of mass $\mu > 0$, in s space dimensions. All arguments k, l, p, q, x, y and all integration variables are to be considered as elements of $\mathbb{R}^s$. As usual,

$\mu(k) = (\mu^2 + k \cdot k)^{\frac{1}{2}}$. We define the Fock space $\mathcal{F}$ as $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$.
 $\varphi \in \mathcal{F}$ means

$$\varphi = \left\{ \varphi^{(n)} \in \mathcal{F}^{(n)}; \|\varphi\|^2 = \sum_{n=0}^{\infty} \|\varphi^{(n)}\|^2 < \infty \right\}.$$

Here, $\mathcal{F}^{(n)}$ is the space of symmetric functions in $L_2(\mathbb{R}^{sn})$. We write the scalar product in $\mathcal{F}^{(n)}$ as

$$(\varphi^{(n)}, \psi^{(n)}) = \int d^s k_1 \dots d^s k_n \overline{\varphi^{(n)}(k_1, \dots, k_n)} \psi^{(n)}(k_1, \dots, k_n).$$

We will write from now on dk instead of $d^s k$.

On $\mathcal{F}$ we have a representation of the canonical commutation relations

$$[a(k), a(l)] = 0 \quad \text{and} \quad [a(k), a^*(k)] = \delta(k-l).$$

The annihilation operator $a(k)$ is defined by

$$\begin{aligned} (a(k) \varphi^{(n)})^{(n-1)}(k_1, \dots, k_{n-1}) &= n^{\frac{1}{2}} \varphi^{(n)}(k, k_1, \dots, k_{n-1}), \\ (a(k) \varphi^{(n)})^{(n')} &= 0 \quad \text{if } n' \neq n-1. \end{aligned}$$

$a^*(k)$ is the adjoint of $a(k)$. The number of particles operator N is defined to be

$$\{N \varphi^{(n)}\} = \{n \varphi^{(n)}\}.$$

An important relation is

$$\|a^{\#}(f)(N+1)^{\frac{1}{2}}\| \leq \|f\|_2, \quad (1)$$

where $\|\cdot\|_2$ denotes the L_2 -norm, and $a^{\#}$ denotes a or a^* , and

$$a^{\#}(f) = \int a^{\#}(k) f(k) dk.$$

We define Wick monomials W by

$$W = \int dk_1 \dots dk_d w(k_1, \dots, k_d) a^*(k_1) \dots a^*(k_i) a(k_{i+1}) \dots a(k_d), \quad (2)$$

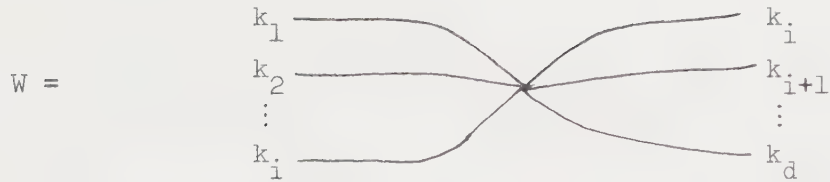
$0 \leq i \leq d$. d is called the degree of W , w is called the (numerical) kernel of W . By convention, a small letter will always denote the numerical kernel of the Wick monomial with the same capital letter.

$\mathcal{D}(A)$ will always denote the natural domain of definition of the operator A on $\mathcal{F}$, i.e. $\mathcal{D}(A) = \{\varphi \in \mathcal{F} \mid \|A\varphi\| < \infty\}$.

LEMMA 1. Let $w \in L_2(\mathbb{R}^{sd})$. Then for every i , $0 \leq i \leq d$, the operator W defined in equation (2) satisfies $\mathcal{D}(W) \supseteq \mathcal{D}((N+1)^{d/2})$ and $\|W(N+1)^{-d/2}\| \leq \text{const.} \|w\|_2$. The constant depends only on d .

This is well-known and will not be proved here.

We suppose that the reader is familiar with Wick's theorem (see e.g. [12]). We will use sometimes a graphical representation described in [3] and in [12]. A Wick-ordered expression of the form (2) is visualized by a vertex of a graph, where the a^* 's are depicted by legs pointing to the left (and going actually to infinity), and the a 's are depicted by legs pointing to the right. So the operator W of equation (2) is drawn as



Wick's theorem is then easily depicted by forming all contraction schemes between the the legs of the factors.

Example: Let

$$W_1 = \text{X} \quad , \quad W_2 = \text{J}$$

Then the Wick ordered expansion of $W_1 W_2$ is

$$W_1 W_2 = : W_1 \underbrace{W_2}_0 : + : W_1 \underbrace{W_2}_1 : + : W_1 \underbrace{W_2}_2 : =$$

$$\text{X J} = \text{X} \text{J} + \text{X} \text{J} + \text{X} \text{J} .$$

The symbol $\underbrace{\quad}_i$ and the graphs imply already the combinatorial factor.

We define finally the time zero free field operator $\phi(x)$ by

$$\phi(x) = \phi^+(x) + \phi^-(x) \quad ,$$

$$\phi^+(x) = \phi^-(x)^* = \int dk \, \mu(k)^{-\frac{1}{2}} \, a^*(k) \, e^{ikx} \quad , \quad x \in \mathbb{R}^S .$$

An important technical tool for constructive quantum field theory, the cutoffs, will appear only in the form of a momentum cutoff σ . We define

$$W_{\sigma} = \int_{|k_j| \leq \sigma, j=1, \dots, d} dk_1 \dots dk_d \, w(k_1, \dots, k_d) \, a^*(k_1) \dots a(k_d) \quad (3)$$

which we will also write as

$$W_{\sigma} = \int dk_1 \dots dk_d \, w_{\sigma}(k_1, \dots, k_d) \, a^*(k_1) \dots a(k_d) \quad .$$

Evidently $W_{\infty} = W$. For reasons explained later, we need no space cutoff.

II.2. Definition and Classification of Persistent Models

Persistent models are theories with two types of particles which are called nucleons and mesons (irrespective of the actual statistics). The main characteristic of the Hamiltonians of such models is that these Hamiltonians commute with the nucleon particle-number operator, i.e. the nucleon number is conserved. The Fock space underlying the theory (which will be defined below) can therefore be decomposed into an infinite direct sum of superselecting sectors with fixed nucleon number. As a further characteristic the Hamiltonian annihilates the Fock vacuum; therefore the free and the physical vacuum are identical. This in turn implies that no space cutoff will be necessary to make the Hamiltonian a well-defined operator (Haag's theorem does not apply), and that one can have therefore a translation invariant interaction. However, the whole inferno of difficulties due to high momenta or to large meson number, known from relativistic quantum field theory, will be present.

We leave now this general context and start the definitions leading to the models which we shall treat below. Both the mesons and the nucleons will be bosons in all the following discussions; many results can be carried over to the case where the nucleon is a fermion.

We define a Fock space $\mathcal{H}$ for the two types of particles:

$$\mathcal{H} = \mathcal{F} \otimes \mathcal{F}$$

Here, $\mathcal{F}$ is the Fock space for one particle type defined in section II.1 . It will be convenient to distinguish several subspaces of $\mathcal{H}$:

$$\mathcal{H}^{(n)} = \mathcal{F}^{(n)} \otimes \mathcal{F} \quad ,$$

the space with fixed nucleon number (often called a sector), and

$$\mathcal{H}^{(n,m)} = \mathcal{F}^{(n)} \otimes \mathcal{F}^{(m)},$$

the space with nucleon and meson number fixed.

We will apply consistently the following conventions:

- $a^\#$ denotes a meson creation or annihilation operator
- $b^\#$ denotes a nucleon creation or annihilation operator
- a or b used as index or as superscript between brackets denotes that the corresponding quantity is used for mesons or nucleons respectively
- n denotes the nucleon number
- m denotes the meson number
- $\omega(k)$ is used for nucleons, $\omega(k) = (\omega^2 + k^2)^{\frac{1}{2}}$, $\omega > 0$
- $\mu(k)$ is used for mesons, $\mu(k) = (\mu^2 + k^2)^{\frac{1}{2}}$, $\mu > 0$
- P is a real polynomial of degree d , d even, P is lower semibounded, $P \neq 0$

We are now prepared to state the main definition:

An interaction is said to be persistent and of type $\mathcal{P}_i$, $i = 1, 2, 3$ if it is of the form $H_i = H_0 + V_i$, with the following properties:

H_0 , the free Hamiltonian is defined by

$$\begin{aligned} H_0 &= H_0^{(a)} + H_0^{(b)} \\ H_0^{(a)} &= \int \mu(k) a^*(k) a(k) dk \\ H_0^{(b)} &= \int \omega(k) b^*(k) b(k) dk \end{aligned}$$

The interaction Hamiltonians V_i are given by the formal expressions

$$\begin{aligned} V_1 &= \int \psi_b^+(x) \psi_b^-(x) \psi_a(x) dx \\ V_2 &= \int \psi_b^+(x) \psi_b^-(x) : \psi_a^2 : (x) dx \end{aligned}$$

$$V_3 = \int \phi_b^+(x) \phi_b^-(x) : P(\phi_a) : (x) dx, \quad \deg P = 4$$

Here P satisfies the general criteria stated above and $P(\phi_a)$ means that the variable of P has been replaced by $\phi_a(x)$.

By construction, H_i maps $\mathcal{H}^{(n)} \cap \mathcal{D}(H_i)$ into $\mathcal{H}^{(n)}$ for each $n = 0, 1, 2, \dots$ so it makes sense to speak of $H_i|_n$, the restriction of H_i to $\mathcal{H}^{(n)}$. We shall write

$$H_i = \bigoplus_{n=0}^{\infty} H_i|_n \quad \text{on} \quad \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \quad (i = 1, 2, 3).$$

Our aims are to give a meaning to these formal Hamiltonians and to derive some of their properties. Before starting this work in the subsequent chapters, we will close this section by the following remarks about these Hamiltonians.

The conclusions of Haag's theorem become irrelevant for persistent models because the physical and the free vacuum coincide. Therefore one can work with models which have no space cutoff. This motivates the preceding definitions. These definitions imply, however, that the numerical kernel of V_i is not an L_2 -function. It is in fact of the form

$$v_i = \delta\left(\sum_{j=1}^{\ell} \pm k_j\right) \tilde{v}_i(k_1, \dots, k_{\ell}),$$

where the reduced kernel $\tilde{v}_i(k_1, \dots, k_{\ell})$ is a continuous function. $\pm k_j$ means $+k_j$ if the variable k_j occurs in an a^* or a b^* and $-k_j$ if it occurs in an a or a b of V_i . All this shows that Lemma 1 is not strong enough to deal with persistent interaction Hamiltonians V_i , $i = 1, 2, 3$. One has the better result:

LEMMA 2. Let W be a Wick monomial of the form

$$W = \int b^*(p_1) b(p_2) a^*(k_1) \dots a^*(k_i) a(k_{i+1}) \dots a(k_d) \cdot \\ \cdot \tilde{w}(p_1, p_2, k_1, \dots, k_d) \delta(p_1 - p_2 + k_1 + \dots + k_i - k_{i+1} - \dots - k_d) dp_1 dp_2 dk_1 \dots dk_d .$$

If
$$\sup_p \int \left| w(p, p + k_1 + \dots + k_i - k_{i+1} - \dots - k_d, k_1, \dots, k_d) \right|^2 dk_1 \dots dk_d = Y$$

is finite then W is defined on $\mathcal{D}(N_b(N_a + 1)^{d/2})$ and

$$\left\| W (N_a + 1)^{-d/2} \right\|_n \leq n Y^{\frac{1}{2}} \cdot \text{const.} , \text{ where the constant depends only on } d .$$

Proof: Let $\varphi \in \mathcal{H}^{(n, m+d-i)}$, $\psi \in \mathcal{H}^{(n, m+i)}$. Let $p + \Sigma \equiv p + k_1 + \dots + k_i - k_{i+1} - \dots - k_d$. Then

$$(\psi, W \varphi) = \left(\frac{(m+i)!}{m!} \frac{(m+d-i)!}{m!} \right)^{\frac{1}{2}} n \cdot \\ \cdot \int \bar{\varphi}(q_1, \dots, q_{n-1}, p; \ell_1, \dots, \ell_m, k_1, \dots, k_i) \psi(q_1, \dots, q_{n-1}, p+\Sigma; \ell_1, \dots, \ell_m, k_{i+1}, \dots, k_d) \cdot \\ \cdot \tilde{w}(p, p+\Sigma, k_1, \dots, k_d) dp dk_1 \dots dk_d dq_1 \dots dq_{n-1} d\ell_1 \dots d\ell_m .$$

The numerical factor can be estimated by

$$n (\dots)^{\frac{1}{2}} \leq n c^d m^{d/2} , \text{ where } c \text{ is a constant}$$

and the integral can be estimated by the Schwartz inequality

$$\int \bar{\varphi} W \varphi \leq \int dp \|\psi\|_2 \|\tilde{w}\|_2 \|\varphi\|_2 , \quad \text{where } \|\cdot\|_2 \text{ means the}$$

L_2 -norm over all variables of the functions except p . Using now

$$\left(\int dp \| \psi \|_2^2 \right)^{\frac{1}{2}} = \| \psi \|_2 \quad \text{and again the Schwartz inequality,}$$

we get

$$\begin{aligned} \int dp \| \psi \|_2 \| \tilde{w} \|_2 \| \varphi \|_2 &\leq \int dp \| \psi \|_2 \| \varphi \|_2 \left(\sup_p \| \tilde{w} \|_2 \right) \leq \\ &\leq \| \psi \|_2 \| \varphi \|_2 \sup_p \| \tilde{w} \|_2, \end{aligned}$$

and this proves the Lemma.

III. CONNECTION WITH OTHER MODEL FIELD THEORIES

As it is our aim to take persistent models as a playground in which we want to explore the difficulties of realistic quantum field models, it may be of interest to see which degree of complexity is already attained by our simple model. One can also ask to what extent persistent models are "nearer" to relativistic quantum field theory than other well-known field models. This comparison will be made now.

III.1. The Scalar Field Model without Recoil

The Hamiltonian of the scalar field model without recoil is given by

$$H = H_0 + H_I ,$$

$$H_0 = m_0 \int dp \, b^*(p) b(p) + \int dq \, \mu(q) a^*(q) a(q) ,$$

$$H_I = \int dp \, dq \, b^*(p+q) b(p) \{ a^*(-q) + a(q) \} \omega(q)^{-\frac{1}{2}} f(q^2) ,$$

where f is the cutoff function. In the literature, the nucleon is mostly taken as a fermion; we have taken a boson for better comparison with the persistent models. Evidently, the corresponding persistent interaction would be of type $\mathcal{P}_1$.

The scalar field model without recoil is known [22] to be soluble in the sense that it is possible to give the exact renormalized one-particle state. The exact S-matrix elements for the scattering of a meson by a nucleon may also be computed and are in fact trivial. Guenin and Velo have also calculated the n -point functions [11].

These encouraging results are more or less a direct consequence of the following particularities of the scalar field model without recoil:

- Let

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}, \text{ then}$$

$$[H_I(t_1), [H_I(t_2), H_I(t_3)]] = 0 \quad (4)$$

- The free and the physical vacuum coincide and one can work in Fock space.

The crucial equation (4) is a direct consequence of the fact that the nucleon is "recoil-less", i.e. its energy does not depend on the momentum. So, as usual with many field models, this beautiful property (4) of the scalar field model without recoil is at the same time its essential weakness.

III.2. The Lee Model

In its standard version (see e.g. [22]), the Lee model describes an interaction between three types of particles which are called usually N , V , and Θ . The process which is described is $V \rightleftharpoons N + \Theta$. Let N_V , N_N and N_Θ denote the corresponding particle number operators. Then it is typical for the Lee model that $N_V + N_N$ and $N_V + N_\Theta$ commute with the Hamiltonian. By stating this as the characteristic point of Lee models, one arrives at a wide class of similar models which may even have more resemblance to relativistic field models than the standard model.

Let n_1 and n_2 be any two positive integers. Let there be two particle types a and b . Then the "generalized" Lee models can be characterized by having a translation invariant Hamiltonian H satisfying

$$[H, n_1 N_a + n_2 N_b] = 0,$$

i.e. one has a sort of charge conservation in these models. The standard Lee model has been discussed rigourously by Schrader[21] and a special case of the generalized Lee model is given by Hepp in [12]. These authors treated the model in the limit of local interaction. We give as an example the interaction treated in [12]:

$$H_0 = \int \omega(k) b^*(k) b(k) dk + \int \mu(k) a^*(k) a(k) dk$$

$$H_I = \int dx \{ \phi_b^+(x) \phi_a^-(x)^2 + \phi_a^+(x) \phi_b^-(x)^2 \}$$

graphically represented as

$$H_I = \text{[diagram 1]} + \text{[diagram 2]}$$

This model has not really a persistent analog but exhibits the same ultra-violet divergencies as some persistent models; it has in sufficiently many space dimensions an infinite mass renormalization. Schrader has shown the existence of a local Hamiltonian by summing up the renormalized Born series for the resolvent. He also discussed scattering in the lowest sectors. For the above version of the Lee model, Hepp has shown the convergence of the Born series with slightly different methods which will be very useful in chapter V.

The Lee model exhibits in all its versions true particle creation; its beauty and its weakness lie in the fact that no iterated application of the Hamiltonian to a state with a finite number of particles will ever lead to a state with infinitely many particles. Free and physical vacua coincide.

III.3. The Nelson Model

This model may be interpreted as a persistent model with a nonrelativistic nucleon. Its Hamiltonian has the form $H = H_O + H_I$, where

$$H_O = (2M)^{-1} \sum_{i=1}^n p_i^2 + \int \mu(k) a^*(k) a(k) dk \quad \text{and}$$

$$H_I = \sum_{i=1}^n \phi_a(x_i) \quad .$$

Here M denotes the nucleon mass and n denotes the (fixed) nucleon number. x_i is the coordinate of the i -th nucleon. The persistent analog of this model is $\mathcal{P}_1$, where

$$H_I = \int \phi_b^+(x) \phi_b^-(x) \phi_a(x) dx \quad .$$

The particle number aspect of the Nelson model is very similar to that of any persistent model and especially to that of the model $\mathcal{P}_1$. The neat commutation behaviour (4) of the scalar field model without recoil is no longer present.

This model has first been discussed by Gross, and others. It was known then that the only divergence of the model was the divergent self-energy of the nucleon (in three space dimensions). Gross [9] found a transformation to split off the infinite part of the Hamiltonian. More precisely, he found a T such that

$$e^T H e^{-T} = H' + E N_b \quad ,$$

with E infinite and such that the self-energy terms coming from H' are finite in all orders of perturbation. Nelson [19] rigorized this construction in showing the existence of a renormalized Hamiltonian (with "local" interaction) as a limit of cutoff versions. This is expressed in the

Theorem: For each $\sigma < \infty$, H_σ is a selfadjoint operator on $\mathcal{H} = \mathcal{F}_b \otimes \mathcal{F}_a$. There is a unique selfadjoint operator $\hat{H}$ on $\mathcal{H}$ such that for all real t and for all ψ in $\mathcal{H}$

$$\lim_{\sigma \rightarrow \infty} e^{-it(H_\sigma - N_b E_\sigma)} \psi = e^{-it\hat{H}} \psi.$$

The operator $\hat{H}$ is bounded below.

Cannon [1] has discussed many further properties of the Nelson model in great detail. He proved important properties of the vacuum expectation values and the existence of a one-particle state (for the nucleons). This will be a partial guide in chapter VI.

Concerning the differences between the Nelson model and the persistent models we note that in the Nelson model, due to the nonrelativistic kinematics, the free energy of the nucleon becomes greater than the lower limit of the continuous spectrum for high momenta. Our second observation leads to more far reaching consequences: Both models conserve the total momentum. One can therefore write for the Nelson model

$$H_\sigma|_1 = \int^\oplus (H)_p dp$$

Then, on $\mathcal{H}^{(1,n)}$,

$$\begin{aligned} (H)_p &= (2M)^{-1} |p - \sum k_i|^2 + \int \mu(k) a^*(k) a(k) dk + \\ &+ \int_{|k| \leq \sigma} \{a(k) + a^*(k)\} \omega(k)^{-\frac{1}{2}} dk \end{aligned}$$

where the k_i are the variables of

$$\varphi^{(1,m)}(p - \sum k_i; k_1, \dots, k_m) \in (\mathcal{H}^{(1,m)})_p.$$

Now, for the corresponding persistent model,

$$(H_I)_p = A_p \phi_a(\delta) A_p dp, \quad ,$$

where A_p is an operator (see chapter IV), and

$$\phi_a(\delta) = \int \phi_a(x) \delta(x) dx.$$

This gives rise to serious problems in semiboundedness and selfadjointness questions.

III.4. Relativistic Field Theories

Relativistic field models offer many more difficulties than the persistent models. In this section we discuss only in what sense the technical advantages of the persistent models are lost when one considers their relativistic counterpart.

It is easy to associate a relativistic interaction with a persistent one. Indeed, to

$$V = \int \phi_b^+(x) : P'(\phi_a) : (x) \phi_b^-(x) dx$$

we associate

$$V_R = \int : \phi_b^2 : (x) : P'(\phi_a) : (x) h(x) dx.$$

Here, P' is any polynomial. The main difference between the two interaction Hamiltonians is the occurrence of a space cutoff $h(x)$ in V_R , which is necessary because V_R contains pure creation terms. Although an argument by Guenin [10] and others makes it clear that this cutoff need not influence the dynamics at small times (if h is constant inside a sphere), this technical

device has disastrous consequences. It is clear that many arguments using momentum conservation must fail for relativistic theories. Another aspect of this fact is that the free and the physical vacuum cannot coincide for relativistic theories, in contrast to the persistent models, where this is always the case.

Persistent models present, however, already the whole scale of problems associated with greatly increasing number of particles and with ultra-violet divergences.

IV. RESULTS FOR CUTOFF HAMILTONIANS

In this chapter we prove semiboundedness and selfadjointness for the momentum cutoff Hamiltonians of persistent models. We treat first the easy case of the interaction $\mathcal{P}_1$, for which the cutoff interaction Hamiltonian is small in the sense of Kato [16] with respect to the free Hamiltonian. The remaining two types of interactions can be treated with the methods of J. Glimm and A. Jaffe [7], [8]. Note that the general results of Lanford, Jaffe and Wightman [17] on cutoff Hamiltonians do not cover the present case because in their treatment the meson self-interaction is always required to dominate the meson-nucleon interaction (e.g. $H_I \sim \phi_a^{2j} + \phi_b^+ \phi_b^- \phi_a^{2j-2}$).

IV.1. The Case of a Small Interaction

In this section we treat the interaction V_1 , which will be seen to be a relatively bounded perturbation of H_0 . Lemma 5 will give in addition conclusions about the selfadjointness of an operator A if $A|_n$ is known to be selfadjoint for all n , and related results.

Let $V_{1,\sigma}$ denote the momentum cutoff operator V_1 (see equation (3)) and let

$$\omega_\sigma^{-1}(k) = \omega^{-1}(k) \text{ if } |k| < \sigma, \text{ zero otherwise.}$$

LEMMA 3. Let $\sigma < \infty$, $n \geq 0$, $\lambda \in \mathbb{R}$. Then $V_{1,\sigma}|_n$ is small in the sense of Kato with respect to $H_0|_n$.

Proof: Note that the reduced numerical kernel of the operator $V_{1,\sigma}$ is an L_2 -function for $\sigma < \infty$, since it is of the form

$$\omega_{\sigma}^{-\frac{1}{2}}(p+q) \omega_{\sigma}^{-\frac{1}{2}}(p) \mu_{\sigma}^{-\frac{1}{2}}(q) \quad .$$

So it remains to consider the meson number problem. Now, by Lemma 2,

$$\begin{aligned} & \left\| V_{1,\sigma} (N_a + 1)^{-\frac{1}{2}} \Big|_n \right\|^2 \leq \\ & \leq n^2 \sup_p \int |\omega_{\sigma}^{-\frac{1}{2}}(p+q) \omega_{\sigma}^{-\frac{1}{2}}(p) \mu_{\sigma}^{-\frac{1}{2}}(q)|^2 dq = n^2 C_{\sigma}^2 \end{aligned}$$

and therefore, for $\varphi^{(n)} \in \mathcal{H}^{(n)} \cap \mathcal{D}(H_0)$,

$$\begin{aligned} \left\| \lambda V_{1,\sigma} \varphi \right\| & \leq n \cdot |\lambda| \cdot C_{\sigma} \left\| (N_a + 1)^{\frac{1}{2}} \varphi \right\| \leq \\ & \leq \varepsilon \left\| H_0 \varphi \right\| + C(n \cdot |\lambda| \cdot C_{\sigma}, \varepsilon) \left\| \varphi \right\| \end{aligned}$$

for any $\varepsilon > 0$. The last inequality above is a well-known relation between N and H_0 (see e.g. [18]), and thus Lemma 3 is proved.

We are now prepared to state the

THEOREM 4. For all $\sigma < \infty$ and all $\lambda \in \mathbb{R}$ and any number of space dimensions, the operator $H_0 + \lambda V_{1,\sigma}$ is selfadjoint on $\mathcal{D}(H_0) \subset \mathcal{H}$.

Proof: The selfadjointness of

$$H|_n = H_0|_n + \lambda V_{1,\sigma}|_n$$

follows by Lemma 3 from the theorem of Kato (Theorem V.4.3. in [16]).

In fact, $\lambda V_{1,\sigma}$ is a symmetric operator by definition and bounded with respect to H_0 by a bound $\varepsilon < 1$, due to Lemma 3.

It remains to show the selfadjointness of $H \equiv \bigoplus_{n=0}^{\infty} H|_n$. We will formulate this in the

LEMMA 5. Let $\mathcal{H}_n$, $n = 1, 2, \dots$ be a family of Hilbert spaces with scalar product $(\cdot, \cdot)_n$ and norm $\|\cdot\|_n$. Let, for every n , A_n be a symmetric (resp. essentially selfadjoint or selfadjoint) operator on $\mathcal{D}_n \equiv \mathcal{D}(A_n) \subset \mathcal{H}_n$. Then $A = \bigoplus_{n=1}^{\infty} A_n$ is symmetric (resp. essentially selfadjoint or selfadjoint) on $\mathcal{D}(A) \subset \mathcal{H} = \{ \varphi \mid \varphi = \{ \varphi_1, \varphi_2, \dots \} ; \varphi_i \in \mathcal{H}_i, i = 1, 2, \dots ; \|\varphi\|^2 \equiv \sum_{n=1}^{\infty} \|\varphi_n\|_n^2 < \infty \}$.

Proof: Note that we define $\mathcal{D}(0)$ always to be the natural domain of definition of the operator 0 ,

$$\mathcal{D}(0) = \{ \varphi \in \mathcal{H} \mid \|0\varphi\| < \infty \}$$

The following remarks are almost trivial but important. $\mathcal{D}(A)$ is dense in $\mathcal{H}$ if the $\mathcal{D}_i$ are dense in the $\mathcal{H}_i$, $i = 1, 2, \dots$. Furthermore, if $\varphi \in \mathcal{D}(A)$, then $\varphi_n \in \mathcal{D}_n$ because

$$\|A\varphi\|^2 = \sum_{n=1}^{\infty} \|A_n \varphi_n\|_n^2.$$

We proceed now to the proof of the different assertions of the theorem.

Symmetry: Let $\varphi, \psi \in \mathcal{D}(A)$. Then

$$\begin{aligned} (\psi, A\varphi) &= \sum_{n=1}^{\infty} (\psi_n, A_n \varphi_n)_n = \sum_{n=1}^{\infty} (A_n \psi_n, \varphi_n)_n = \\ &= (A\psi, \varphi) = (A^* \psi, \varphi). \end{aligned}$$

The second equality is due to the symmetry of A_n , the last equality is the definition $(\psi, A\varphi) = (A^* \psi, \varphi)$, and $(\psi, A\varphi) = (A\psi, \varphi)$ means $A \subset A^*$ as we intended to show.

Selfadjointness: We show $A^* \subset A$. Let $\varphi \in \mathcal{D}(A^*)$; $(\varphi, A\psi)$ is defined for all $\psi \in \mathcal{D}(A)$ and there is a χ such that $(\varphi, A\psi) = (\chi, \psi)$. $\varphi \in \mathcal{D}(A^*)$ means also that $(\varphi_n, A_n \psi_n)_n$ is defined for all $\psi_n \in \mathcal{D}_n$; so $\varphi_n \in \mathcal{D}((A_n)^*) = \mathcal{D}_n$ and by hypothesis $(\varphi_n, A_n \psi_n)_n = (A_n \varphi_n, \psi_n)_n$. So $\chi = \{A_n \varphi_n\} = A\varphi$; i.e. $\varphi \in \mathcal{D}(A)$ and $A^* \subset A$.

Essential selfadjointness: Prove selfadjointness of A^* as before of A .

This completes the proof of Lemma 5.

The case of the interaction V_2 in one space dimension without cutoff but for small coupling constant will be treated in chapter VI, Corollary 25b. With cutoff but with arbitrary real coupling constant, this model will be treated in the next section.

IV.2. Large Positive Perturbations

In this section we will describe results for the models $\mathcal{P}_2$ and $\mathcal{P}_3$ for $s = 1$ space dimension. For these models, the meson number part of the interaction is not smaller than the free Hamiltonian. This makes an application of Kato's theorem impossible. Problems similar to those of $\mathcal{P}_2$ and $\mathcal{P}_3$ have, however, been solved by Glimm and Jaffe for pure boson selfinteractions. With persistent models, new difficulties arise because of the presence of two types of particles.

The main result will be Theorem 12, which asserts selfadjointness of $H_0 + V_\sigma$ ($V_\sigma = V_{2,\sigma}$ or $V_{3,\sigma}$ (degree of the polynomial = 4)). We will prove this by the following steps: Theorem 6 shows essential selfadjointness of V_0 on a domain $\mathcal{D}_0$. This follows from a modification (Lemma 7) of a result of Glimm and Jaffe [7] (stated here as Lemma 8), combined with some manipulations on unbounded operators (Lemma 9 and 10). Theorem 11 shows the semiboundedness of the Hamiltonian. These results and the estimates of Lemma 13 allow to apply the singular perturbation theory of Glimm and Jaffe [8] (stated here as Lemma 14), which will in turn prove Theorem 12.

THEOREM 6: Let P be a real polynomial. Then, for $\sigma < \infty$ and fixed n ,

$$V_\sigma|_n \equiv \phi_{b,\sigma}^+(x) \phi_{b,\sigma}^-(x) : P(\phi_{a,\sigma}) : (x) d^1x \Big|_n \text{ is essentially selfadjoint on } \mathcal{D}_{0,n} \equiv \bigcap_{p=\sigma}^{\infty} \mathcal{D}(H_0^p|_n) .$$

Proof: The idea is to use known results about the essential selfadjointness of $f : P(\phi_a) : (x) h(x) dx$ ($h \in \mathcal{F}$), on a meson Fock space. We shall see indeed that by a unitary transformation U the meson-part of the persistent interaction can be written in a more manageable form. The nucleon interaction can then be viewed, due to the momentum coservation by V_σ , as a multiplicative

perturbation of the meson part. Before we motivate the course of the proof any further, we introduce the transformation U mentioned above.

Let $\mathcal{H}$ denote as usual the Fock space for mesons and nucleons. Let U be a unitary transformation on $\mathcal{H}$:

If $\varphi^{(n,m)} \in \mathcal{H}^{(n,m)}$ then

$$\begin{aligned} (U \varphi^{(n,m)}) (p; k_1, \dots, k_{n-1}; q_1, \dots, q_m) &= \\ &= \varphi^{(n,m)} (k_1, \dots, k_{n-1}, p - \sum_{j=1}^{n-1} k_j - \sum_{j=1}^m q_j; q_1, \dots, q_m) . \end{aligned}$$

We define $\tilde{a}^*(k)$ and $\tilde{a}(k)$ on $U \mathcal{H}$ by

$$\begin{aligned} \left(\int dk \tilde{a}^*(k) f(k) U \varphi^{(n,m)} \right) (p; k_1, \dots, k_{n-1}; q_1, \dots, q_{m+1}) &= \\ = (m+1)^{-\frac{1}{2}} \sum_{j=1}^{m+1} f(q_j) (U \varphi^{(n,m)}) (p; k_1, \dots, k_{n-1}; q_1, \dots, q_j, \dots, q_{m+1}) & \end{aligned}$$

and

$$\begin{aligned} \left(\int dk \tilde{a}(k) f(k) U \varphi^{(n,m)} \right) (p; k_1, \dots, k_{n-1}; q_1, \dots, q_{m-1}) &= \\ = m^{-\frac{1}{2}} \sum_{j=1}^m \int f(r) & \\ (U \varphi^{(n,m)}) (p; k_1, \dots, k_{n-1}; q_1, \dots, q_{j-1}, r, q_j, \dots, q_{m-1}) dr . & \end{aligned}$$

For fixed $p, k_1, \dots, k_{n-1}$, the form of these operators is identical with that of the usual meson creation and annihilation operators. This was, of course, the motivation for the choice of the transformation U . Note that

$$U \tilde{a}^*(k) U^{-1} \neq \tilde{a}^*(k) .$$

A straightforward calculation shows now that

$$\begin{aligned} & \left(\int \phi_{b,\sigma}^+(x) \phi_{b,\sigma}^-(x) : P(\phi_{a,\sigma}) : (x) dx \right) \psi^{(n,m)} = \\ & = A_{\sigma}^2 U^{-1} \int : P(\phi_{\tilde{a},\sigma}) : (x) \delta(x) dx U \psi^{(n,m)}, \end{aligned}$$

where

$$A_{\sigma}|_n = n^{-\frac{1}{2}} \int b^*(k) \omega_{\sigma}(k)^{-\frac{1}{2}} b(k) dk.$$

An analogic expression is

$$\begin{aligned} & \left(\int \phi_{b,\sigma}^+(x) : P(\phi_{a,\sigma}) : (x) \phi_{b,\sigma}^-(x) dx \right) \psi^{(n,m)} = \\ & = A_{\sigma} U^{-1} \int : P(\phi_{\tilde{a},\sigma}) : (x) \delta(x) dx U A_{\sigma} \psi^{(n,m)}. \end{aligned}$$

We are now prepared to continue the outline of the proof of Theorem 6. Define $\mathcal{K}^{(n)} = U \mathcal{H}^{(n)}$; $\mathcal{K}^{(n)}$ can be viewed as $L_2(\mathbb{R}^n) \otimes \mathcal{F}_a$, where $\mathcal{F}_a$ denotes the meson Fock space. Then

$$: P(\phi_{\tilde{a},\sigma}) : (\delta) = \mathbf{1} \otimes : P(\phi_{a,\sigma}) : (\delta).$$

The proof of Theorem 6 will proceed by the following steps: Let

$$\mathcal{D}_a \equiv \bigcap_{k=0}^{\infty} \mathcal{D}((H_O^{(a)})^k) \subset \mathcal{F}_a,$$

$: P(\phi_{a,\sigma}) : (\delta)$ is essentially selfadjoint on $\mathcal{D}_a$ (Lemma 7),
 $: P(\phi_{\tilde{a},\sigma}) : (\delta)$ is essentially selfadjoint on $\mathcal{D}' \times \mathcal{D}_a$, where $\mathcal{D}'$ is dense in $L_2(\mathbb{R}^n)$ (Lemma 9). $\mathcal{D}'$ is chosen in such a way that $U^{-1}(\mathcal{D}' \times \mathcal{D}_a) \subset \mathcal{D}_0$. We complete the proof of Theorem 6 by showing that $A_{\sigma}^2 U^{-1} : P(\phi_{\tilde{a},\sigma}) : (\delta) U$ is essentially selfadjoint on $U^{-1}(\mathcal{D}' \times \mathcal{D}_a) \subset \mathcal{D}_0$.

(Lemma 10).

LEMMA 7. Let P be a real polynomial. Then $T = P(\Phi_\sigma) : (\delta)$ is essentially selfadjoint on $\mathcal{D}_a \equiv \bigcap_{q=1}^{\infty} \mathcal{D}((H_0^{(a)})^q)$.

Proof: We use a modification of the proof which Glimm and Jaffe [7] gave for the case $T = P(\Phi) : (h)$, $h \in \mathcal{F}$. The combination of their Lemma 3.3 and 3.4 gives

LEMMA 8. [7] Let $\mathcal{M}$ denote the (maximal abelian) von Neumann algebra of bounded operators generated by the spectral projections of all $t = 0$ fields $\phi(f)$, $f \in \mathcal{F}(\mathbb{R}^1)$. Let T be any operator with domain $\mathcal{D}_1$ such that

- 1) $T \mathcal{D}_1 \subset \mathcal{D}(\phi(f)^k)$ for all $f \in \mathcal{F}$, k
- 2) $T \mathcal{D}_1 \subset \mathcal{D}(T|_{\mathcal{D}_1}^*)$
- 3) $[T, \phi(f)^k] \mathcal{D}_1 = 0$ for all $f \in \mathcal{F}$, k
- 4) T is a symmetric operator

Then T is essentially selfadjoint on $\mathcal{D}_1$.

(A closed symmetric operator commuting with the elements of a maximal abelian von Neumann algebra is selfadjoint).

We will now prove Lemma 7 by verifying the hypotheses of Lemma 8 for the operator $T = P(\Phi_\sigma) : (\delta)$, and $\mathcal{D}_1$ the domain obtained by applying any polynomial of the free fields $\phi_a(f) = \phi(f)$, $f \in \mathcal{F}$ to the vacuum (meson Fock space only).

Let d be the degree of P . Then for any h_σ satisfying

$$\tilde{h}_\sigma(k) = \begin{cases} 1 & \text{if } |k| \leq (d+1)\sigma \\ 0 & \text{if } |k| > (d+2)\sigma \end{cases}, \quad \tilde{h}_\sigma \in \mathcal{F}(\mathbb{R}^1),$$

the expression $: P(\Phi_\sigma) : (\delta)$ is identical with $: P(\Phi_\sigma) : (h_\sigma)$ as can easily be seen by going to momentum space. ($\sim$ denotes Fourier transform).

We verify now the hypotheses of Lemma 8, one by one.

- 1): This is true for $T = : P(\Phi_a) : (h_\sigma)$ and therefore also for $: P(\Phi_{a,\sigma}) : (\delta) = : P(\Phi_{a,\sigma}) : (h_\sigma)$.
- 2): Follows by the symmetry of T .
- 3): Trivial; $[\Phi_{a,\sigma}(x), \Phi_a(y)] = 0$ (time zero free fields!).
- 4): By definition of T .

So the conclusions of Lemma 8 hold and thus Lemma 7 is proved, because

$$\mathcal{D}_1 \subset \mathcal{D}_a.$$

Now we prove that $: P(\Phi_{a,\sigma}) : (\delta) = 1 \otimes : P(\Phi_{a,\sigma}) : (\delta)$ is essentially selfadjoint on the domain $\mathcal{D}' \times \mathcal{D}_a$.

LEMMA 9. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let B be an essentially selfadjoint operator on $\mathcal{D}_2 \subset \mathcal{H}_2$. Then for every $\mathcal{D}_1 \subset \mathcal{H}_1$ which is dense in $\mathcal{H}_1$, $1 \times B$ is essentially selfadjoint on $\mathcal{D}_1 \times \mathcal{D}_2$.*

Proof: That B is essentially selfadjoint on $\mathcal{D}_2$ is equivalent to saying that for some nonreal ζ , $(B - \zeta 1)\mathcal{D}_2$ is dense in $\mathcal{H}_2$ (see e.g. [16], Problem V.3.17, page 272). We want to show that

$$((1 \otimes B) - \zeta(1 \otimes 1))\mathcal{D}_1 \times \mathcal{D}_2$$

is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. But

* $\mathcal{D}_1 \times \mathcal{D}_2$ denotes the set of all finite linear combinations of vectors in $\mathcal{D}_1$ times vectors in $\mathcal{D}_2$.

$$1 \otimes B - \zeta(1 \otimes 1) = 1 \otimes (B - \zeta 1) \quad \text{and so}$$

$$1 \otimes (B - \zeta 1) (\mathcal{D}_1 \times \mathcal{D}_2) = \mathcal{D}_1 \times (B - \zeta 1) \mathcal{D}_2 = \mathcal{D}_1 \times \mathcal{D}_2'$$

which is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$ because $\mathcal{D}_1$ and $\mathcal{D}_2'$ are dense in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. So Lemma 9 is proved.

To show that $U^{-1} : P(\Phi_{\mathbb{R}, \sigma}) : (\delta) U$ is essentially selfadjoint on $\mathcal{D}_0$ it suffices to exhibit a dense domain $\mathcal{D}' \subset L_2(\mathbb{R}^n)$ such that $U^{-1}(\mathcal{D}' \times \mathcal{D}_a)$ is contained in $\mathcal{D}_0$. Now

$$U \mathcal{D}_0 = \mathcal{D}_3 \equiv \bigcap_{k=1}^{\infty} \mathcal{D}((U H_0 U^{-1})^k),$$

but $U H_0^{(a)} U^{-1} = H_0^{(a)}$, so $\mathcal{D}' \times \mathcal{D}_a \subset \mathcal{D}_3$ if only the functions in $\mathcal{D}'$ are decreasing fast enough.

It remains to consider the perturbation by A_0 .

LEMMA 10. Let T be essentially selfadjoint on $\mathcal{D} \subset \mathcal{H}$. Let S be a bounded selfadjoint operator and let T commute with S , $ST \subseteq TS$. Then STS is essentially selfadjoint on $\mathcal{D}$.

Proof: We show first that $(SST)^* = (SST)^{**}$, i.e. that SST is essentially selfadjoint on $\mathcal{D}(SST) = \mathcal{D}(T) \equiv \mathcal{D}$.

We use first $TS \supseteq ST$, $T^* = T^{**}$, $S^* = S$ and prove $(ST)^* = (ST)^{**}$. $TS \supseteq ST$ implies $(ST)^* \supseteq (TS)^* \supseteq ST^*$ where the last inclusion is a well-known relation between any two operators ([16], Problem III.5.26., page 168. Note also that $(BU)^* = U^*B^*$ if B is bounded). By this last remark, $(ST)^* = T^*S$ and we have then $T^*S \supseteq ST^*$. Therefore $(T^*S)^* \subseteq (ST^*)^* = T^{**}S = T^*S$. So finally $(T^*S)^* = T^*S = (ST)^* = (ST)^{**}$ and our first goal is reached.

Now we use $(ST)S \supseteq S(ST)$, $(ST)^* = (ST)^{**}$, $S^* = S$ and the same manipulations as for the first conclusion will furnish $(S(ST))^* = (S(ST))^{**}$, in other words SST is essentially selfadjoint.

We prove now $(STS)^* = (SST)^*$, which is seen as follows: Use $STS \supseteq SST$ so $(STS)^* \subseteq (SST)^* = (SST)^{**} \subseteq (STS)^{**}$. But STS is symmetric, so $(STS)^{**} \subseteq (STS)^*$, so $(STS)^* = (SST)^* = (STS)^{**}$. The fact that STS is essentially selfadjoint on $\mathcal{D}$ follows then from $STS \supseteq SST$ and from the fact that SST is essentially selfadjoint on $\mathcal{D}$. This completes the proof of Lemma 10.

Theorem 6 follows now by Lemma 10 by taking

$$\begin{aligned} S &= A_{\sigma,n}, \text{ and} \\ T &= U : P(\phi_{a,\sigma}) : (\delta) U^{-1} \Big|_n, \text{ and} \\ \mathcal{D} &= \mathcal{D}_0|_n \subset \mathcal{H}^{(n)}. \end{aligned}$$

THEOREM 11. Let V_σ be as in Theorem 6. For $\sigma < \infty$ and fixed n , $H_0 + V_\sigma$ is lower semibounded if the polynomial P is.

Proof: Write again

$$V_\sigma = A_\sigma U : P(\phi_{a,\sigma}) : (\delta) U^{-1} A_\sigma.$$

It is known (Nelson [19], or Glimm [5, Lemma 5.2]) that $:P(\phi_\sigma): (h_\sigma)$ is lower semibounded if the polynomial P is lower semibounded and if h_σ is nonnegative and in $\mathcal{F}(\mathbb{R}^1)$. The bound is, of course, not independent of σ , so that $:P(\phi_\sigma): (\delta) = :P(\phi_\sigma): (h_\sigma) \geq -C_\sigma$. Because of the selfadjointness of A_σ , $A_{\sigma,n} : P(\phi_\sigma) : (\delta) A_{\sigma,n} \geq -C_\sigma \omega(0)^{-1}$. H_0 is also semibounded since it is positive and so Theorem 11 is proved.

We state now the main theorem of this section:

THEOREM 12. Let P be a real polynomial of degree $d = 2$ or $d = 4$ with positive leading coefficient. For $\sigma < \infty$ and any n ,

$$H_n = H_0|_n + H_{I,\sigma}|_n = H_0|_n + \int \phi_{b,\sigma}^+(x) \phi_{b,\sigma}^-(x) : P(\phi_{a,\sigma}) : (x) dx|_n$$

is selfadjoint with domain $\mathcal{D}_n = \mathcal{D}(H|_n) = \mathcal{D}(H_0|_n) \cap \mathcal{D}(H_{I,\sigma}|_n)$ and essentially selfadjoint on $\mathcal{D}_{0,n} = \bigcap_{k=0}^{\infty} \mathcal{D}(H_0^k|_n)$.

Note that $\mathcal{D}(H_0|_n) \cap \mathcal{D}(H_{I,\sigma}|_n) \neq \{0\}$.

Remark: L. Rosen [20] has proved selfadjointness of $(H_0 + : P(\Phi) : (h))_{s=1}$ for any polynomial P of even degree with positive leading coefficient (and h nonnegative). His methods are different from those used by Glimm and Jaffe. It is therefore not possible to generalize Theorem 12 directly to the case of an arbitrary semibounded polynomial.

Proof of Theorem 12: This proof is derived from an argument in [8]; the estimates which are necessary to verify the hypotheses of the theorem given there are similar to those given in [7] for the case of a pure boson self-interaction. We give these estimates in

LEMMA 13. For any $\epsilon > 0$ and for any fixed sector n there is a constant b such that (as bilinear forms)

$$- [H_0^{\frac{1}{2}}, [H_0^{\frac{1}{2}}, H_{I,\sigma}]] \leq \epsilon H_0^2 + b \quad \text{and} \quad (5)$$

$$- [N_a, [N_a, H_{I,\sigma}]] \leq \epsilon N_a^3 + b \quad \text{.} \quad (6)$$

Proof of Lemma 13: We start with an estimate on a vector Θ which is a n nucleon, m meson state. Then

$$\begin{aligned} & \left(\left[\left(H_O^{(a)} + H_O^{(b)} \right)^{\frac{1}{2}}, a(k) \right] \Theta \right) (\ell_1, \dots, \ell_n, k_1, \dots, k_{m-1}) = \\ & = \lambda(k, \ell_i, k_j) a(k) \Theta. \end{aligned}$$

$$\text{Now } \lambda = \left(\sum_{i=1}^{m-1} \mu(k_i) + \sum_{i=1}^n \omega(\ell_i) \right)^{\frac{1}{2}} - \left(\sum_{i=1}^{m-1} \mu(k_i) + \sum_{i=1}^n \omega(\ell_i) + \mu(k) \right)^{\frac{1}{2}},$$

and therefore

$$\begin{aligned} |\lambda| & \leq \mu(k)^{\frac{1}{2}} & \text{and also} \\ |\lambda| & \leq \frac{1}{2} \mu(k) \left(\sum_{i=1}^{m-1} \mu(k_i) + \sum_{i=1}^n \omega(\ell_i) \right)^{-\frac{1}{2}} \leq (2 \omega(0))^{-\frac{1}{2}} \mu(k) (m+n-1)^{-\frac{1}{2}}. \end{aligned}$$

A similar estimate holds for

$$\left(\left[\left(H_O^{(a)} + H_O^{(b)} \right)^{\frac{1}{2}}, a^*(k) \right] \Theta \right) = \lambda' a^*(k) \Theta,$$

$$|\lambda'| \leq \mu(k)^{\frac{1}{2}},$$

$$|\lambda'| \leq (2 \omega(0))^{-\frac{1}{2}} \mu(k) (m+n)^{-\frac{1}{2}}.$$

Let $h_{I,\sigma}$ be the numerical kernel of a Wick ordered term of $H_{I,\sigma}$;

$$h_{I,\sigma}(q_1, \dots, q_4, p) = \omega(p)^{-\frac{1}{2}} \omega(p + \sum \pm q_i)^{-\frac{1}{2}} \prod_{i=1}^4 \mu(q_i)^{-\frac{1}{2}}.$$

It follows then from the above estimates that there is a $\hat{h}_{I,\sigma}$,

$$|\hat{h}_{I,\sigma}| \leq \text{const.} \left| \sum_{i=1}^4 \mu(q_i) \cdot h_{I,\sigma} \right|$$

such that

$$\begin{aligned} & \left\| \left[H_O^{\frac{1}{2}}, \left[H_O^{\frac{1}{2}}, H_{I,\sigma} \right] \right] (N+I)^{-2} \right\|^2 \leq \\ & \leq \text{const.} \sup_p \int |\hat{h}_{I,\sigma}(q_1, \dots, q_4, p)|^2 dq_1 \dots dq_4 \quad \text{on a fixed sector.} \end{aligned}$$

Equation (5) is now a direct consequence of the fact that

$$N \leq \varepsilon H_0^2 + \text{const.} \quad \text{for any } \varepsilon > 0.$$

(6) is a direct consequence of the fact that $H_{I,\sigma}$ is at most of fourth order in the mesons. This completes the proof of Lemma 13.

We are now prepared to prove Theorem 12 by means of the following theorem of Glimm and Jaffe:

LEMMA 14 [8, Theorem 8].

Let A, B , and N be (unbounded) operators, satisfying

- 0) A, B , and N are selfadjoint.
- 1) A and N commute.
- 2) As bilinear forms, $A \geq N \geq 0$.
- 3) $\mathcal{D} \equiv \bigcap_{k=0}^{\infty} \mathcal{D}(A^k)$ is a core for B (see e.g. [16] for the def. of "core").
- 4) As bilinear forms on $\mathcal{D} \times \mathcal{D}$, $0 \leq aN + B + \text{const.}$, for some $0 < a \leq \frac{1}{2}$.
- 5) $0 \leq \varepsilon A^2 + \text{const. } B + [A^{\frac{1}{2}}, [A^{\frac{1}{2}}, B]] + \text{const.}$, for some $\varepsilon > 0$, $2a + \varepsilon < 1$.
- 6) B is a bounded operator from $\mathcal{H}_\nu$ to $\mathcal{H}_{-\nu}$ and from $\mathcal{H}_\alpha$ to $\mathcal{H}_\beta$ for some α, β, ν ; $\beta > 0$. ($\mathcal{H}_\nu$ is the Hilbert space with the norm $\|\psi\|_\nu^2 = (N^{\nu/2} \psi, N^{\nu/2} \psi)$, where $(\cdot, \cdot)$ is the usual scalar product).
- 7) If $\nu \geq 2$ then, for some $\eta > \nu - 2$, as bilinear form,

$$0 \leq \varepsilon N^{(\eta+2)} + [N^{(\eta+1)/2}, [N^{(\eta+1)/2}, B]] + \text{const.}$$

Then $A + B$ is selfadjoint.

We apply now this lemma with

$$\mathcal{H} = \mathcal{H}^{(n)}, \quad N = N_a, \quad A = H_0|_n, \quad B = \bar{H}_{I,\sigma}|_n, \quad v = 2, \quad \eta = 1.$$

We verify the hypotheses of Lemma 14 one by one:

0) H_0 is selfadjoint, N_a is selfadjoint (well-known).

$H_{I,\sigma}$ is essentially selfadjoint on $\mathcal{D}_0$ (by Theorem 6).

1), 2) are trivial.

3) is true by Theorem 6.

4) is true by Theorem 11.

5) is the equation (5) of Lemma 13.

6) holds because $H_{I,\sigma}$ is a sum of Wick-ordered operators of degree ≤ 4 in the meson operators.

7) is the equation (6) of Lemma 13.

This proves the selfadjointness of $H|_n$ asserted in Theorem 12. The essential selfadjointness is derived from the knowledge that $H|_n$ is selfadjoint (see Theorem 4.1. in [7]). The proof given there uses only the commutation properties of H_0 , $H_{I,\sigma}$ etc., and the fact that $H_{I,\sigma} (N + I)^{-2}$ is a bounded operator; it can therefore be taken over for our case. This completes the proof of Theorem 12.

V. CONVERGENCE OF THE BORN SERIES FOR THE MODEL $H_0 + \int \phi_b^+(x) \phi_b^-(x) \phi_a(x) dx$

IN THREE SPACE DIMENSIONS

V.1. General Remarks

This model needs an infinite mass renormalization. Our first aim is to construct a symmetric operator on a dense domain by adding an appropriate counterterm to the formal Hamiltonian (section 2). We will then establish an estimate for the Born series which is uniform in the momentum cutoff σ . This is done by a resummation of the renormalized Born series for finite σ (section 3). The last section of this chapter covers the selfadjointness problem and the relation between the symmetric operator constructed in section 2 and the selfadjoint operator obtained from the converging Born series.

V.2. Existence of a Symmetric Operator in the Limit $\sigma \rightarrow \infty$

We consider a fixed sector with n nucleons, $n > 0$, and we shall always write A instead of $A|_n$, $\mathcal{H}$ instead of $\mathcal{H}^{(n)}$ etc. . Let σ denote the usual momentum cutoff. Our aim is to construct a symmetric operator H_σ such that with a mass renormalization M_σ one has on a dense domain $\mathcal{D} \subset \mathcal{H}$

$$\text{s-lim}_{\sigma \rightarrow \infty} (H_0 + V_\sigma - M_\sigma) T_\sigma = H_\infty T_\infty \quad \text{and}$$

$$\text{s-lim}_{\sigma \rightarrow \infty} T_\sigma = T_\infty \quad \text{and}$$

$T_\infty \mathcal{D}$ dense in $\mathcal{H}$. The procedure we shall apply is essentially the same as the one described by Hepp [12], and initiated by Glimm [6].

Let

$$H_0 = \int \mu(k) a^*(k) a(k) d^3k + \int \omega(k) b^*(k) b(k) d^3k \quad (7a)$$

$$V_\sigma = V_\sigma^c + V_\sigma^a \quad (7b)$$

$$V_\sigma^c = \int_{\substack{|k_i| \leq \sigma \\ i=1,2,3}} \omega(k_1)^{-\frac{1}{2}} \omega(k_2)^{-\frac{1}{2}} \mu(k_3)^{-\frac{1}{2}} \delta(k_1 - k_2 + k_3) \cdot b^*(k_1) b(k_2) a^*(k_3) d^3k_1 d^3k_2 d^3k_3 \quad (7c)$$

$$V_\sigma^a = (V_\sigma^c)^* = \int_{\substack{|k_i| \leq \sigma \\ i=1,2,3}} \omega(k_1)^{-\frac{1}{2}} \omega(k_2)^{-\frac{1}{2}} \mu(k_3)^{-\frac{1}{2}} \delta(k_1 - k_2 - k_3) \cdot b^*(k_1) b(k_2) a(k_3) d^3k_1 d^3k_2 d^3k_3 \quad (7d)$$

The superscripts c and a stand for creation and annihilation (of the meson) respectively.

We give now some important estimates on norms which will form the principal tool throughout this chapter.

LEMMA 15. Let r^c be a continuous function. Let

$$R^c = \int r^c(k_1, k_2, k_3) \delta(k_1 - k_2 + k_3) b^*(k_1) b(k_2) a^*(k_3) dk_1 dk_2 dk_3.$$

Then, for all $\varphi \in \mathcal{D}(N_b (N_a + 1)^{\frac{1}{2}})$,

$$\|R^c \varphi\|^2 \leq \|N_b (N_a + 1)^{\frac{1}{2}} \varphi\|^2 \sup_p \int |r^c(q, p, p-q)|^2 dq.$$

Proof: This lemma is a special case of Lemma 2 for $d = 1$.

We have also the adjoint version of Lemma 15:

LEMMA 16. Let r^a be a continuous function. Let

$$R^a = \int r^a(k_1, k_2, k_3) \delta(k_1 - k_2 - k_3) b^*(k_1) b(k_2) a(k_3) dk_1 dk_2 dk_3 .$$

Then, for all $\psi \in \mathcal{D}(N_b (N_a + 1)^{\frac{1}{2}})$,

$$\| R^a \psi \|^2 \leq N \| N_b (N_a + 1)^{\frac{1}{2}} \psi \|^2 \sup_q \int |r^a(p, q, p-q)|^2 dp .$$

Proof: As Lemma 15.

An important estimate relating H_0 and V is the following

LEMMA 17. Let $n > 0$ be fixed, $\tau \geq \frac{1}{2}$, R^a as in Lemma 16. Then

$$\| R^a (H_0^{(a)} + 1)^{-\tau} |_n \|^2 \leq n^2 \int dp \mu(p)^{-2\tau} \sup_q |r^a(p, q, p-q)|^2 .$$

Proof: This lemma has first been given by Glimm [4 , Lemma 2.4.1.], with a slight technical error in the proof. A correct and extended proof can be found in [14 , Lemma 50 (page 137)].

Remark: The main point of Lemma 17 is that $H_0^{-\frac{1}{2}-\epsilon}$, $\epsilon > 0$, compensates at the same time the unboundedness of the annihilation operator and gives an improvement in momentum behaviour. The adjoint of Lemma 17 holds as well.

This ends our estimates.

The construction of a dense domain for the renormalized Hamiltonian of a theory of type $\mathcal{P}_1$ follows a pattern described by Friedrichs [3], in a form similar to the form given in [12]. We shall see later (Theorem 24) that the symmetric operator which is constructed on a dense domain is in fact essentially selfadjoint. The practical calculations show that this property does not depend too much on the several possible variants for the choice of the dense domain. We will therefore restrict ourselves to a very simple choice. It is only for the discussion of the one-particle structure by Friedrich's methods that a more sophisticated dressing transformation would be needed.

The dense domain for the symmetric operator is constructed by giving a dressing transformation which is "almost intertwining" between H and H_0 .

We start this construction by defining the Γ -operation of Glimm: Let

$$W = \int w(p_1, p_2, q_1, \dots, q_d) b^*(p_1) b(p_2) a^*(q_1) \dots a^*(q_i) a(q_{i+1}) \dots a(q_d) \cdot dp_1 dp_2 dq_1 \dots dq_d .$$

Then we define

$$\Gamma W = \int w(p_1, p_2, q_1, \dots, q_d) (\omega(p_1) + \mu(q_1) + \dots + \mu(q_i))^{-1} \cdot b^*(p_1) b(p_2) a^*(q_1) \dots a^*(q_i) a(q_{i+1}) \dots a(q_d) dp_1 dp_2 dq_1 \dots dq_d .$$

We will sometimes denote the numerical kernel of ΓW by γw . In short, the above definition means that W is "divided" by the energy of the created particles.

Let now T_σ be a formal solution of the equation

$$T_\sigma = 1 - \Gamma(V_\sigma^c T_\sigma) .$$

Then T is "almost intertwining" $H = H_0 + V_\sigma^c$ and H_0 in the sense that

$$(H_0 + V_\sigma^c) T_\sigma = : T_\sigma H_0 : .$$

T_σ is formally given by

$$T_\sigma = \sum_{k=0}^{\infty} T_{\sigma,k} ; \quad T_{\sigma,0} = 1 ,$$

$$T_{\sigma,k} = - \Gamma(V_\sigma^c T_{\sigma,(k-1)}) = (-1)^k \underbrace{\Gamma(V_\sigma^c \Gamma(V_\sigma^c \dots \Gamma(V_\sigma^c) \dots))}_{k \text{ times } V_\sigma^c} , \quad k=1,2,\dots$$

Let finally

$$\mathcal{D}_n = \{ \varphi \in \mathcal{H}^{(n)} \mid \varphi = \{ \varphi^{(n,0)}, \varphi^{(n,1)}, \dots \} , \varphi^{(n,m)} \neq 0$$

only for finitely many m , $\varphi^{(n,m)}$ in the space $\mathcal{D}$ of L.Schwartz $\}$.

We start to control these formal manipulations by

LEMMA 18. For $\sigma \leq \infty$, $T_\sigma|_n$ is defined on $\mathcal{D}_n$ and for all $\varphi \in \mathcal{D}_n$,

$$\lim_{\sigma \rightarrow \infty} T_\sigma \varphi \quad \text{exists. } T_\sigma \text{ is invertible.}$$

Remark: Compared with other field models, the point of Lemma 18 is that we need no truncation to show convergence of the series for T . This is due to the fact that there is only one meson creator in V_σ^c and due to some careful estimates on V_σ^c .

Proof: We state some estimates on the operation Γ . Let W be of the form

$$W = \int b^*(k_0) a^*(k_1) \dots a^*(k_p) b(k_0 + \dots + k_p) dk_0 \dots dk_p w(k_0, \dots, k_p) , \quad p \geq 1 ,$$

and let W be defined on $\varphi = \varphi^{(n,m)}$.

Then

$$\| \Gamma W \varphi^{(n,m)} \| \leq (m+p) p^{-1} \| (H_0^{(a)})^{-1} W \varphi^{(n,m)} \| ,$$

as is evident from the definition. Using now Lemma 17 and writing

$$W = \int dk W(k) a^*(k) , \text{ we find for any } \varepsilon < \frac{1}{2} ,$$

$$\begin{aligned} \| \Gamma W \varphi^{(n,m)} \|^2 &\leq ((m+p) p^{-1} \| (H_0^{(a)})^{-1} W \varphi^{(n,m)} \|)^2 \\ &\leq \left(\frac{(m+p)}{p} \left(\frac{1}{\mu(0)p} \right)^{\frac{1}{2}-\varepsilon} \| (H_0^{(a)})^{-\frac{1}{2}-\varepsilon} W \varphi^{(n,m)} \| \right)^2 \\ &\leq \text{const.} \left(\frac{m+p}{p^{3/2-\varepsilon}} \right)^2 \int dk \mu(k)^{-1-2\varepsilon} \| W(k) \varphi^{(n,m)} \|^2 \end{aligned}$$

(8)

The conclusion is that Γ is regularizing in momentum behaviour and in particle number behaviour.

We apply now this information to prove the existence of T_σ . Note that $T_{\sigma,k}|_n$ is a sum of products of at most n Wick-ordered operators: Let $S_1, \dots, S_n$ denote the Wick-ordered fragments of one term $T'_{\sigma,k-1}$ of $T_{\sigma,k-1}$. Then

$$(V_\sigma^c T'_{\sigma,k-1}) = : (V_\sigma^c \underbrace{S_1 \dots S_n}_o) : + : (V_\sigma^c \underbrace{S_1 \dots S_n}_\wedge) : .$$

The term $: (V_\sigma^c \underbrace{S_1 \dots S_n}_o) :$ is zero on $\mathcal{H}^{(n)}$ unless at least for one i , $1 \leq i \leq n$, $S_i = 1$.

We claim that

$$\| T_{\sigma,k} \varphi^{(n,m)} \|^2 \leq \left(\frac{(m+k)!}{m! k! 3/2-\varepsilon} \right)^2 n^{2k} D_{\sigma,\varepsilon}^k \| \varphi^{(n,m)} \|^2 , \quad k \geq 1.$$

Proof: By induction: First we consider $k = 1$. There, one has

$$\begin{aligned} \|T_{\sigma,1} \varphi^{(n,m)}\|^2 &= \|\Gamma(V_{\sigma}^c) \varphi^{(n,m)}\|^2 = \\ &= (m+1) n^2 \int \sup_p \mu(q)^{-1-2\epsilon} \omega(p)^{-1} \omega(p-q)^{-1} \mu(q)^{-1} dq \|\varphi^{(n,m)}\|^2 \end{aligned}$$

by (8) and Lemma 17. The $\int \sup$ is finite $= D_{\sigma,\epsilon}$.

Now we assume that the result is correct for $1 \leq k \leq j-1$ and we want to conclude for j . If the j -th V_{σ}^c is not contracted to one of the Wick-ordered fragments of $T_{\sigma,j-1}$, then one takes the product of the norms and the numerical factor coming from the Γ -operation. Otherwise $T_{\sigma,j}$ is of the form

$$T_{\sigma,j} = - \int a^*(p) T_{\sigma,j-1}(p) dp$$

and by (8)

$$\begin{aligned} \|T_{\sigma,j} \varphi^{(n,m)}\|^2 &\leq \\ &\leq ((m+j) j^{-3/2+\epsilon})^2 (n-1)^2 \int \mu(p)^{-1-2\epsilon} \|T_{\sigma,j-1}(p)\|^2 dp \|\varphi^{(n,m)}\|^2. \end{aligned}$$

Suppose for simplicity that $T_{\sigma,j-1}$ consists of exactly one connected component (the other cases are treated similarly), with q creating meson legs:

$$T_{\sigma,j-1} = \int b^*(k_0) a^*(k_1) \dots a^*(k_q) b(k_0 + \dots + k_q) t(k_0, \dots, k_q) dk_0 \dots dk_q.$$

Then

$$\begin{aligned} T_{\sigma,j-1}(p) &= \int b^*(k_0 - p) a^*(k_1) \dots a^*(k_q) b(k_0 + \dots + k_q) \cdot \\ &\cdot t(k_0, \dots, k_q) \omega^{-\frac{1}{2}}(k_0 - p) \omega^{-\frac{1}{2}}(k_0) \mu^{-\frac{1}{2}}(p) dk_0 \dots dk_q. \end{aligned}$$

By direct computation one gets that

$$\| T_{\sigma, j-1}(p) \varphi \|^2 \leq \mu(p)^{-1} \sup_{k_0} \omega(k_0 - p)^{-1} \omega(k_0)^{-1} \| T_{\sigma, j-1} \varphi \|^2 .$$

Using

$$\sup_{k_0} \omega(k_0 - p)^{-1} \omega(k_0)^{-1} \leq \text{const.} \cdot \omega(p)^{-1} ,$$

one gets finally

$$\| T_{\sigma, j} \varphi^{(n, m)} \|^2 \leq ((m+j) j^{-3/2+\varepsilon} n)^2 D_{\sigma, \varepsilon} \| T_{\sigma, j-1} \varphi^{(n, m)} \|^2 ,$$

and this proves the assertion. As an immediate consequence we have

$$\begin{aligned} \| T_{\sigma} \varphi \|^2 &\leq (\text{const.})^k (C_{\varphi} + k)!^2 C_{\varphi}!^{-2} k!^{-3+2\varepsilon} \| \varphi \|^2 \\ &\leq (\text{const.}(\varphi))^k k!^{-1+2\varepsilon} \| \varphi \|^2 . \end{aligned}$$

Here $C_{\varphi} = \max \{m \mid \varphi^{(n, m)} \neq 0\}$, which is finite for all $\varphi \in \mathcal{D}_n$.

The existence of the limit $\sigma \rightarrow \infty$ follows from the monotony of D_{σ} and the boundedness of D_{∞} , and the invertibility of T_{σ} , $\sigma \leq \infty$, follows from the fact that T_{σ} increases the number of particles. This completes the proof of Lemma 18.

We calculate now the "intertwining error" which is due to the fact that T is only an approximately intertwining operator. To make this intertwining error at all defined, it is necessary to introduce an infinite renormalization. We have to consider

$$(H_0 + V_{\sigma}^c) T_{\sigma} = :T_{\sigma} H_0: \quad \text{and} \quad V_{\sigma}^a T_{\sigma} . \quad (9)$$

$:T_{\sigma} H_{\sigma}:$ is defined on $\mathcal{D}_n$ for $\sigma \neq \infty$, however $V_{\sigma}^a T$ contains in its Wick-ordered expansion contractions which give rise to infinities as $\sigma \rightarrow \infty$. These are exactly all the terms of the form

$$V_{\sigma}^a \underbrace{\Gamma(V_{\sigma}^c \Gamma(V_{\sigma}^c \dots \Gamma(V_{\sigma}^c \Gamma V^c) \dots))}_2$$

where $\underbrace{\quad}_2$ means as usual twice contracted. One is thus led to introduce a mass renormalization M_{σ}

$$M_{\sigma} = V_{\sigma}^a \underbrace{\Gamma(V_{\sigma}^c)}_2 \quad (9)$$

which ensures that

$$V_{\sigma}^a \underbrace{\Gamma(V_{\sigma}^c \Gamma(V_{\sigma}^c \dots \Gamma V_{\sigma}^c \dots))}_2 = V_{\sigma}^a \underbrace{\Gamma(V_{\sigma}^c)}_2 \cdot \Gamma(V_{\sigma}^c \dots \Gamma V_{\sigma}^c \dots) \quad (10)$$

converges strongly as $\sigma \rightarrow \infty$. The proof of this fact is well-known in many variants and can be found in [12]. Formula (10) shows in fact that

$$\lambda^2 M_{\sigma} = \lambda^2 V_{\sigma}^a \underbrace{\Gamma V_{\sigma}^c}_2$$

is a good choice for a counter term to $H_{\sigma} + \lambda V_{\sigma}^a + \lambda V_{\sigma}^c$. We define therefore the total renormalized Hamiltonian for the model $\mathcal{P}_1$ in three space dimensions as

$$H_{\text{ren},\sigma} = H_0 + \lambda V_{\sigma}^a + \lambda V_{\sigma}^c + \lambda^2 M_{\sigma}.$$

Then one has

THEOREM 19. For $\varphi \in \mathcal{D}_n$, $n = 1, 2, \dots$

$$\text{s-lim}_{\sigma \rightarrow \infty} H_{\text{ren},\sigma} T_{\sigma} \varphi = H_{\infty} T_{\infty} \varphi$$

exists and defines a symmetric operator H_{∞} on $T_{\infty} \mathcal{D}$.

Proof: We will not give the proof of Theorem 19. Let us remark that such theorems are well-known. Theorem 19 is more or less a specialization of Theorem 4.1 in [12] (which treats the Yukawa interaction). In contrast to the Yukawa interaction case we need no truncation for the definition of T_0 and therefore Lemma 4.4 in [12] is not even needed for our case.

V.3. Convergence of the Renormalized Born Series

A convenient way to study the selfadjointness of an operator which is in its cutoff version a small perturbation of H_0 is to look at the Born series

$$\begin{aligned} R_\sigma(z) &= (z - H_{\text{ren},\sigma})^{-1} \\ &= R_0(z) \sum_{k=0}^{\infty} \left\{ (\lambda V_\sigma + \lambda^2 M_\sigma) R_0(z) \right\}^k \end{aligned} \quad (11)$$

where $R_0(z) = (z - H_0)^{-1}$; V_σ and M_σ are defined in equations (7) and (9) respectively.

Schrader [21] has shown the convergence of the Born series for the Lee model which has the same momentum divergence as the persistent model $\mathcal{P}_1$ in three dimensions. Hepp has given in [12] a somewhat more elegant proof for a similar model. If his estimates are refined by the Lemmata 15 to 17 given above, they can be applied to the model we are considering now.

The two principal ideas for the convergence proof of the Born series are a resummation (according to the powers of the coupling constant λ) and the fact that the norm of the k -th order term can be estimated by a product of norms. The absolute convergence of the series will justify a posteriori the resummation.

We shall consider from now on a fixed sector $n \geq 1$. We state now the main result of this section.

THEOREM 20. The renormalized Born series (11) for $R_\sigma(z)$ converges (if properly reordered) in norm for all z satisfying $\text{Re } z < -\delta(\lambda, n)$, uniformly in $\sigma \leq \infty$.

Proof: The resummation mentionned above is done as follows. We write

$$R_{\sigma}(z) = \sum_{k=0}^{\infty} \lambda^k \sum_{i_1, \dots, i_k=1}^2 (R_0(z) \underbrace{V_{i_1}}_1 R_0(z) \dots V_{i_k} R_0(z))_{\text{ren}} \quad (12)$$

where $V_1 = V_{\sigma}^a$, $V_2 = V_{\sigma}^c$ and $(\quad)_{\text{ren}}$ means that every factor $V_1 R_0(z) V_2$ is replaced by

$$: V_1 \underbrace{R_0(z)}_0 V_2 : + : V_1 \underbrace{R_0(z)}_1 V_2 : + (V_1 \underbrace{R_0(z)}_2 V_2)_{\text{ren}}$$

with

$$(V_1 \underbrace{R_0(z)}_2 V_2)_{\text{ren}} = V_1 \underbrace{R_0(z)}_2 V_2 - M_{\sigma}.$$

Here, R_0 is contracted with V_1 and V_2 .

This resummation allows a factoring of the corresponding norms which is expressed in the following estimates.

LEMMA 21. Let $0 < \varepsilon < \frac{1}{4}$. For $\text{Re } z \leq -1$ there exist constants C_n and C_{ε} (independent of $0 < \sigma \leq \infty$) such that

$$\|R_0(z)\| \leq \text{const. dist}(z, [0, \infty))^{-1} \quad (13)$$

$$\|V_{\sigma}^a R_0(z)^{(1+\varepsilon)/2}|_n\| \leq C_{\varepsilon} C_n \quad (14)$$

$$\|R_0(z)^{(1+\varepsilon)/2} V_{\sigma}^c|_n\| \leq C_{\varepsilon} C_n \quad (15)$$

$$\|R_0(z)^{\varepsilon/2} V_{\sigma}^a \underbrace{R_0(z)}_0 V_{\sigma}^c R_0(z)^{\varepsilon/2}|_n\| \leq C_{\varepsilon}^2 C_n^2 \quad (16)$$

$$\|R_0(z)^{\varepsilon/2} V_{\sigma}^a \underbrace{R_0(z)}_1 V_{\sigma}^c R_0(z)^{\varepsilon/2}|_n\| \leq C_{\varepsilon}^2 C_n^2 \quad (17)$$

$$\|R_0(z) (V_{\sigma}^a \underbrace{R_0(z)}_2 V_{\sigma}^c)_{\text{ren}}|_n\| \leq C_{\varepsilon}^2 C_n^2 |z|^{-1+\varepsilon} \quad (18)$$

$$\|V_{\sigma}^a R_0(z) (V_{\sigma}^a \underbrace{R_0(z)}_2 V_{\sigma}^c)_{\text{ren}} R_0(z) V_{\sigma}^c|_n\| \leq C_{\varepsilon}^4 C_n^4 |z|^{-1+2\varepsilon} \quad (19)$$

Proof of Lemma 21: In principle, Lemma 21 is the same as formulas 3.38 to 3.43 of Theorem 3.3 in [12]. However, in some situations, the infinite meson number problem which is typical for the persistent model but absent in the Lee model has to be taken care of. This is done by improving the estimates of Hepp in the sense of our Lemmata 15 to 17: H_O^{-1} does not only furnish an additional negative power but it also compensates one meson creation or annihilation operator.

We give the necessary additional remarks for each of the formulae (13) to (19).

(13): This is well-known (see e.g. [16]).

(14): We apply Lemma 17

$$\begin{aligned} & \| V_\sigma^a (z - H_O)^{-(1-\epsilon)/2} \|^2 \leq \| V_\sigma^a (H_O^{(a)} + 1)^{-(1-\epsilon)/2} \|^2 \leq \\ & \leq n^2 \int dp \mu(p)^{-1-\epsilon} \sup_q \omega(p)^{-1} \omega(q)^{-1} \mu(p-q)^{-1} . \end{aligned}$$

But

$$\sup_q \omega(q)^{-1} \mu(p-q)^{-1} \leq \text{const. } \omega(p)^{-1} ,$$

thus the integral converges and (14) follows.

(15): This is the adjoint of (14).

(16): To obtain (16), we write

$$\begin{aligned} & R_O^{\epsilon/2} : V_\sigma^a \underbrace{R_O}_{\circ} V_\sigma^c : R_O^{\epsilon/2} = \\ & = : V_\sigma^a \underbrace{R_O^{1+\epsilon}}_{\circ} V_\sigma^c : + : [R_O^{\epsilon/2}, V_\sigma^a] \underbrace{R_O^{1+\epsilon/2}}_{\circ} V_\sigma^c : + \\ & + : V_\sigma^a \underbrace{R_O^{1+\epsilon/2}}_{\circ} [V_\sigma^c, R_O^{\epsilon/2}] : + : [R_O^{\epsilon/2}, V_\sigma^a] \underbrace{R_O}_{\circ} [V_\sigma^c, R_O^{\epsilon/2}] : \end{aligned}$$

and (16) can easily be obtained by estimates of the type of (14) and (15).

(17): This is obtained in the same manner as (16).

(18) and (19): These two relations are not trivial. They follow, however, directly from the proof given by Hepp for the analogical case of the Lee model, because the two meson legs of V^a and V^c are contracted. This means that the meson number problem does not arise and pure energy estimates - as given by Hepp - are sufficient.

This completes the proof of Lemma 21.

The rest of the proof of Theorem 20 is now identical to the proof of Theorem 3.3 in [12], which consists in splitting appropriately the powers of R_0 in order to write each term of the Born series as a product of terms occurring inside of one of the norms of Lemma 21. It is evident that this is always possible. One finds then that

$$\left\| (R_0(z) V_i R_0(z) \dots R_0(z) V_i R_0(z))_{\text{ren}} \right\|_n \leq C_\epsilon^k C_n^k |z|^{-\tau k} 3^{k/2}, \quad (20)$$

for some $\tau > 0$. The factor $3^{k/2}$ comes from the definition of $(\)_{\text{ren}}$. One gets then

$$\| R_0(z) \| \leq K^k |z|^{-\tau k} |\lambda|^k \quad \text{with} \quad K = 2 \cdot 3^{\frac{1}{2}} C_\epsilon C_n, \quad \text{Re } z < -1$$

and this proves Theorem 20.

Remark: The proof can also be done in another way by estimating

- a) the norm of each Wick-ordered contribution of order k (after resummation) by the methods used for any superrenormalizable theory [2] and
- b) by estimating the number of such terms.

We do not present this proof here because much more space would be needed than for the version given above.

V.4. Selfadjointness of the Hamiltonian

In the preceding section we have accumulated enough information on the model

$$H_0 + \int \psi_b^+(x) \psi_b^-(x) \psi_c(x) d^3x$$

so that we can state now several strong results. The convergence of the Born series is indeed a very practical property:

We can construct a selfadjoint operator H from the resolvent, and what is even better, we can show that the symmetric operator H_∞ constructed in section V.2. is essentially selfadjoint. Its closure coincides with H .

We start with the construction of H from R_∞ which is due to Schrader [21]. Note that $R_\sigma(z)|_n$ is, for appropriate z , the resolvent of the selfadjoint operator

$$H_\sigma|_n = H_0 + V_{1,\sigma}|_n + M_\sigma|_n$$

if $\sigma < \infty$ (see Theorem 4). Then we have

THEOREM 22. Let $R_\infty(z)$ denote the sum

$$\lim_{\sigma \rightarrow \infty} (R_0(z) \sum_{k=0}^{\infty} [(\lambda V_{1,\sigma} + \lambda^2 M_\sigma) R_0(z)]^k)_{\text{ren}}$$

of Theorem 20. Let $H_n = z - R(z)^{-1}|_n$. $H|_n$ is a selfadjoint, linear operator, bounded below.

Proof: Schrader [21], Corollary 1, Lemma 5, Lemma 6 and Theorem 4.

We will now establish the relation between the operator H_∞ defined by Theorem 19 and the operator H defined by Theorem 22. The two theorems which we will state below are tailored like Theorem 3.4 and 3.5 in [12]. There are, however, some differences in the proofs which are typical for the new degree of difficulty introduced by the persistent model (infinite number of particles in each sector) with respect to the Lee model (bounded number of particles in each sector).

THEOREM 25. Let $H|_n$ be the selfadjoint operator defined by Theorem 22. Let $H_\infty|_n$ be the symmetric operator defined by Theorem 19. For $n = 0, 1, 2, \dots$ one has $H|_n \supseteq H_\infty|_n$.

Proof: This is essentially Theorem 3.4 of [12]. Let, as before,

$$\mathcal{D} = \left\{ \varphi \in \mathcal{H} \mid \varphi^{(n,m)} = 0 \text{ for } m \text{ large enough, } \varphi^{(n,m)} \text{ in the space of Schwartz} \right\}.$$

If $\sigma < \infty$ then $\mathcal{D}(H_\sigma) = \mathcal{D}(H_0)$ by Theorem 4. Because H_σ is defined on $T_\sigma \mathcal{D}$, we have $T_\sigma \mathcal{D} \subseteq \mathcal{D}(H_0)$, and also

$$\text{s-lim}_{\sigma \rightarrow \infty} R_\sigma(z)^{-1} T_\sigma \varphi = (z - H_\infty) T_\infty \varphi$$

for all $\varphi \in \mathcal{D}$. On the other hand

$$\text{s-lim}_{\sigma \rightarrow \infty} R_\sigma(z) R_\sigma(z)^{-1} T_\sigma \varphi = R_\infty(z) (z - H_\infty) T_\infty \varphi = T_\infty \varphi$$

and by definition $\mathcal{D}(H_\infty) = T_\infty \mathcal{D}$, so that $\mathcal{D}(H_\infty) = T_\infty \mathcal{D} \subset \mathcal{D}(H)$ because H is defined on $R_\infty \mathcal{H}$. The fact that the operators H and H_∞ act identically on their common domain $\mathcal{D}(H_\infty)$ follows directly from their construction (equation 3.59 in [12]).

Let $(A)^{-}$ denote the closure of the (closable) operator A .

THEOREM 24. $H_{\infty}|_n$ is essentially selfadjoint on $\mathcal{D}_n$. Its closure coincides with H ; $(H_{\infty})^{-} = H$.

Remark: For the Lee model, one has, by construction

$$\mathcal{D}((H_{\infty})^{-}) = T_{\infty} \mathcal{D}(H_0) .$$

This can no longer be true for the persistent model in question because $\mathcal{D}(T_{\infty}) \not\subset \mathcal{D}(H_0)$. We use nevertheless methods which differ only slightly from those given in [12].

Proof: We know already (Theorem 23) that $H_{\infty} \subseteq H$ and therefore $(H_{\infty})^{-} \subseteq (H)^{-}$. But $(H)^{-} = H$ because a selfadjoint operator is closed, and therefore $(H_{\infty})^{-} \subseteq H$.

It remains therefore to prove $\mathcal{D}(H) \subset \mathcal{D}((H_{\infty})^{-})$. Now $\mathcal{D}(H_{\infty}) = T_{\infty} \mathcal{D}$ and $\mathcal{D}((H_{\infty} T_{\infty})^{-}) = \mathcal{D}(:T_{\infty} H_0:)$ because the intertwining error is a small perturbation of $:T_{\infty} H_0:$. The inclusion $\mathcal{D}(H) \subset \mathcal{D}((H_{\infty})^{-})$ will therefore follow from

$$R_{\infty} \mathcal{H} \subset T_{\infty} \mathcal{D}(:T_{\infty} H_0:) \text{ or from } T_{\infty}^{-1} R_{\infty} \mathcal{H} \subset \mathcal{D}(:T_{\infty} H_0:).$$

This means that the theorem is proved if we can show that

$$:T_{\infty} H_0: T_{\infty}^{-1} R_{\infty}$$

is a bounded operator. One finds by simple manipulations that

$$:T_{\infty} H_0: T_{\infty}^{-1} R_{\infty} = z R_{\infty} - 1 - \left(\sum_{k=0}^{\infty} V^a R_0(z) ((V^a + V^c) R_0(z))^k \right)_{\text{ren}} .$$

The boundedness of the $(\quad)_{\text{ren}}$ term follows from the estimates of Lemma 21, with an obvious change (omission of the lefthand $R_o^{\varepsilon/2}$) in inequalities (16) and (17). This completes the proof of Theorem 24.

VI. SPECTRAL PROPERTIES

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VI.1. Semiboundedness of $H_0 + \int \phi_b^+(x) \phi_b^-(x) : \phi_a^2 : (x) dx$

The object of this section is to exhibit a beautiful example of how the nucleon part of the persistent interaction takes over the rôle of the space cutoff $h(x)$ in the corresponding boson-selfinteraction $\int : \phi^2 : (x) h(x) dx$. For this interaction, Glimm has proved semiboundedness [5] by "completing the square". We can follow (at least in a formal sense) exactly his arguments. We will give further results on this interaction in two corollaries.

THEOREM 25. Let $H_0 = H_0^{(a)} + H_0^{(b)}$. Then

$$H_0 + \int \phi_b^+(x) \phi_b^-(x) : \phi_a^2 : (x) dx$$

is a semibounded operator in any sector n for one space dimension.

Remark: The bound is not uniform in n .

Proof: Let

$$\phi(k) = \int e^{ikx} \phi_a(x) dx = \phi^*(-k) ,$$

$$\phi^+(k) = \int e^{ikx} \phi_b^+(x) dx = (\phi^-)^*(-k) .$$

We define

$$A(\eta) = \int \phi^-(-\ell - \eta) \phi(\ell) d\ell$$

$$A^*(\eta) = \int \phi^+(-k + \eta) \phi(k) dk .$$

Then we can write

$$\begin{aligned}
 & \int \phi_b^+(x) \phi_b^-(x) : \phi_a^2 : (x) dx = \\
 &= \int \phi^+(p) \phi^-(q) : \phi(k) \phi(\ell) : \delta(p+q+k+\ell) dp dq dk d\ell = \\
 &= \int \phi^+(-k + \eta) \phi^-(-\ell - \eta) : \phi(k) \phi(\ell) : d\eta dk d\ell = \\
 &= \int : A^*(\eta) A(\eta) : d\eta = X
 \end{aligned}$$

Let now

$$B_\rho = \int_{|\eta| \leq \rho} : A^*(\eta) A(\eta) : d\eta \quad \text{and let } X = B_\rho + B'_\rho.$$

Then we assert (as in [5]) that B_ρ is semibounded and that B'_ρ is small relatively to H_0 (in each sector). It is, in fact, even small with respect to N_a .

The relative smallness of B'_ρ is an immediate consequence of the cutoff: B'_ρ is the sum of four terms of the form

$$B = \int b^*(k_1) b(-k_1 \pm k_2 \pm k_3) a^{\#}(k_2) a^{\#}(k_3) v(k_1, k_2, k_3) dk_1 dk_2 dk_3,$$

each of which is bounded by $\varepsilon(\rho) (N_a + 1)$ because

$$\begin{aligned}
 & \left\| (N_a + 1)^{-\frac{1}{2}} B (N_a + 1)^{-\frac{1}{2}} \right\|^2 \leq \varepsilon(\rho) = \\
 &= \text{const.} \sup_{|\eta| \geq \rho} \int \omega(-k+\eta)^{-1} \omega(-\ell-\eta)^{-1} \mu(k)^{-1} \mu(\ell)^{-1} dk d\ell \quad (*)
 \end{aligned}$$

by Lemma 2. The constant depends only on n and $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$.

To prove the semiboundedness of B_ρ , we will undo the Wick-ordering.

But here

$$\begin{aligned} & \int_{|\eta| \leq \rho} : A^*(\eta) A(\eta) : d\eta = \\ & = \int_{|\eta| \leq \rho} A^*(\eta) A(\eta) d\eta - \int_{|\eta| \leq \rho} d\eta \int dk \omega(k)^{-1} \phi^+(-k + \eta) \phi^- (+k + \eta). \end{aligned}$$

The first term of the right hand side is obviously positive and the second is smaller than $H_0^{(b)} + c(\rho)1$, because it is, for fixed η , of the form

$$b^*(\ell) b(\ell) v_\eta(\ell) d\ell \quad \text{where} \quad v_\eta \in L_2.$$

This concludes the proof of Theorem 25.

This proof implies even the stronger

COROLLARY 25a. For any $\varepsilon > 0$,

$$\varepsilon N_a + \int \phi_b^+(x) \phi_b^-(x) : \phi_a^2 : (x) dx$$

is a semibounded operator in any sector n for one space dimension.

This was also the result proved by Glimm in [5] for the pure quadratic boson-selfinteraction. In fact Theorem 25 alone is redundant, because the interaction V_2 in one dimension is a small perturbation of H_0 . This is seen by the same estimates which lead to equation (*) in the proof of Theorem 25. We have therefore the

COROLLARY 25b. In one space dimension, and for small enough real λ ,

$$H_0 + \lambda \int \phi_b^+(x) \phi_b^-(x) : \phi_a^2 : (x) dx$$

is a selfadjoint operator on $\mathcal{D}(H_0)$.

Proof: The $\varepsilon(\rho)$ calculated in the proof of Theorem 25 tends not only to zero as $\rho \rightarrow \infty$, it is also finite for $\rho \neq 0$. Using Lemma 2, we write

$$\begin{aligned} & \left\| \int \Phi_b^+(x) \Phi_b^-(x) : \Phi_a^2 : (x) dx (N_a + 1)^{-1} \right\|^2 \leq \\ & \leq 4 \cdot \sup_p \int \omega(p)^{-1} \omega(p-q-r)^{-1} \mu(q)^{-1} \mu(r)^{-1} dq dr \cdot \text{const.} \end{aligned}$$

Corollary 25a follows then for small enough coupling constant λ in the same manner as the conclusions of Theorem 4 were derived.

VI.2. Particle Structure for the Model $H_0 + \int \phi_b^+(x) \phi_b^-(x) \phi_a(x) dx$

In this section, we come back to the model treated extensively in chapter V, where we discussed the convergence of the Born series. By refining somewhat our estimates we shall show now the existence of a one-nucleon state for the same model. (The physical one-meson state is in fact identical to the free one-meson state).

The result presented is not completely satisfactory, because the methods used do not allow to get a precise information about the physical spectrum. It would indeed better if the one-nucleon state could be expressed in terms of Friedrichs' perturbation theory [3] ; see also [12]. Partly, this program works: The estimates on the graphs of the Friedrichs expansion for the one-nucleon state are almost identical to those used for the Born series. (Let Γ denote Friedrichs' Γ -operation, $[H_0, \Gamma W] = W$. Then $\Gamma = R_0(\omega(p))$ on operators W which annihilate exactly one nucleon and contain otherwise only creation operators). The additional difficulties of Friedrichs' perturbation theory come from the infinitely many finite mass renormalizations (which are necessary for the definition of the one-nucleon state and which provide for a physical one-nucleon spectrum $\omega(p)$). So far it is not clear if the Friedrichs one-particle expansion is really convergent or not.

Note that for persistent interactions no invariance arguments for the spectrum apply.

As the interaction conserves the total momentum, we shall use the total momentum decomposition of the space $\mathcal{H}^{(1)}$. We have the conventions

$$\mathcal{H}^{(1)} = \int^{\oplus} \mathcal{H}_p^{(1)} dp, \quad \mathcal{H}_p^{(1)} \cong \mathcal{H}^{(0)}.$$

Let $\varphi_p^{(1,m)} \in \mathcal{H}_p^{(1)}$. The isomorphism is given by

$$\begin{aligned}
\varphi_p^{(1,m)}(k_1, \dots, k_m) &= \varphi^{(1,m)}(p - \sum_{i=1}^m k_i; k_1, \dots, k_m) \\
(\varphi_p^{(1,m)}, \varphi_p^{(1,m)})_p &\equiv (\varphi^{(1,m)}, \varphi^{(1,m)})_p = \\
&= \int dk_1 \dots dk_m \varphi^{(1,m)}(p - \sum_{i=1}^m k_i; k_1, \dots, k_m) \varphi^{(1,m)}(p - \sum_{i=1}^m k_i; k_1, \dots, k_m).
\end{aligned}$$

Furthermore we define

$$\|\varphi\|_p^2 = (\varphi, \varphi)_p, \quad \|\varphi\|^2 = \int dp (\varphi, \varphi)_p.$$

For a discussion of consistency of these definitions, see Cannon [1], chapter IV.2.; for a general discussion, see Neimark and Fomin [17].

Note that on the vacuum φ_0 ,

$$\|b^*(p)\varphi_0\|_p = 1, \quad \|b^*(p)\varphi_0\|_q = 0 \text{ if } p \neq q.$$

This is consistent with

$$\left\| \int f(p') b^*(p') dp' \varphi_0 \right\|_p = f(p) \quad \text{for continuous } f$$

and therefore with

$$\left\| \int f(p') b^*(p') dp' \varphi_0 \right\|^2 = \int |f(p)|^2 dp.$$

Finally, we shall consider the reduced operators: $(A)_p$ is the reduction of $A|_1$ to the total momentum p . This ends the notational introduction.

The spectrum of H_0 takes the form

$$\text{spec } (H_0)_p = \{ \omega(p), [\Omega(p), \infty) \} ,$$

where

$$\Omega(p) = \min_q (\omega(p-q) + \mu(q)) = \sqrt{p^2 + (\omega(0) + \mu(0))^2} .$$

Note that this spectral behaviour differs from the one encountered in the Nelson model, where $(H_0)_p$ has only an isolated eigenvalue for small enough $|p|$. In contrast to the Nelson model, the model $\mathcal{P}_1$ allows therefore a one-nucleon state with arbitrarily high (but bounded) momentum, provided the coupling constant is small enough. The precise statement of this fact will be given in Theorem 29.

The results sketched above will rely on the analytic perturbation theory of Kato [16], more or less in the form described in [1].

LEMMA 26. Let R be given, $0 \leq R < \infty$. Let $\varepsilon = (\Omega(R) - \omega(R))/2$.

Then there exists a $\lambda_0 = \lambda_0(R, \omega, \mu) > 0$ such that for all $|p| \leq R$, for all $|\lambda| \leq \lambda_0$, and for z such that $|z - \omega(p)| = \varepsilon$, the expression $R_\lambda(z) b^*(p) \varphi_0$ exists as a vector in $\mathcal{H}_p^{(1)}$. Here,

$$R_\lambda(z) = \sum_{k=0}^{\infty} \lambda^k (R_0(z) (V R_0(z))^k)_{\text{ren}} , \quad V = V^a + V^c + M .$$

(see also (12)).

Remark: The point, and also the difficulty of the lemma lies in the fact that z is now near the spectrum of H_0 . Convergence has to be restored therefore by other means than by negative powers of $|z|$, namely by the boundedness of the total momentum ($|p| \leq R$) and by the smallness of λ .

Proof: Let W be a Wick-ordered operator of the form

$$W = \int b^*(p - \sum_{i=1}^d k_i) a^*(k_1) \dots a^*(k_d) b(p) w(p, k_1, \dots, k_d) dk_1 \dots dk_d. \quad (21)$$

We state the main point of the proof of Lemma 26 as

LEMMA 27. Let $0 < R < \infty$, let φ_0 be the vacuum, $\varepsilon = (\Omega(R) - \omega(R))/2$. For $|z - \omega(p)| = \varepsilon$, there exists a finite constant $C(R)$, such that for all Wick monomials of the form (21) the inequality

$$\| (H_0 - z)^{-1} W b^*(p) \varphi_0 \|_p \leq C(R) \| (H_0 + 1)^{-1} W b^*(p) \varphi_0 \|_p$$

holds. $C(R)$ is independent of d .

Proof of Lemma 27: Because W is of the form (21), we can write

$$\begin{aligned} & ((H_0 - z)^{-1} W b^*(p) \varphi_0) (q, k_1, \dots, k_d) = \\ & = (\omega(q) + \sum_{i=1}^d \mu(k_i) - \omega(p) + n\varepsilon)^{-1} w(q, k_1, \dots, k_d) \delta(p - q - \sum_{i=1}^d k_i) \end{aligned}$$

with $|n| = 1$. Therefore

$$\begin{aligned} & \| (H_0 - z)^{-1} W b^*(p) \varphi_0 \|_p^2 = \\ & = \int dk_1 \dots dk_d \left| \left(\omega(p - \sum_{i=1}^d k_i) + \sum_{i=1}^d \mu(k_i) - \omega(p) + n\varepsilon \right)^{-1} w(p - \sum_{i=1}^d k_i, k_1, \dots, k_d) \right|^2 \end{aligned}$$

and similarly

$$\begin{aligned} & \| (H_0 + 1)^{-1} W b^*(p) \varphi_0 \|_p^2 = \\ & = \int dk_1 \dots dk_d \left| \left(\omega(p - \sum_{i=1}^d k_i) + \sum_{i=1}^d \mu(k_i) + 1 \right)^{-1} w(p - \sum_{i=1}^d k_i, k_1, \dots, k_d) \right|^2. \end{aligned}$$

Therefore $C(R)$ is determined by

$$C(R) = \sup_{\substack{d > 0 \\ |p| \leq R \\ k_i; |n| = 1}} \left| \frac{(\omega(p - \sum_{i=1}^d k_i) + \sum_{i=1}^d \mu(k_i) + 1)}{(\omega(p - \sum_{i=1}^d k_i) + \sum_{i=1}^d \mu(k_i) - \omega(p) + n\varepsilon)} \right|$$

$$= 2 (\omega(R) + 1) \cdot (\Omega(R) - \omega(R))^{-1}.$$

$C(R)$ is indeed independent of d and therefore Lemma 27 is proved.

We continue now the proof of Lemma 26. This is now very easy. If z and p satisfy the hypotheses of Lemma 27, then by Lemma 27,

$$\begin{aligned} & \left\| \lambda^k (R_0(z) - (V R_0(z))^k)_{\text{ren}} b^*(p) \varphi_0 \right\|_p \leq \\ & \leq \left\| \lambda^k C(R)^k (R_0(-1) - (V R_0(-1))^k)_{\text{ren}} b^*(p) \varphi_0 \right\|_p \leq \\ & \leq |\lambda|^k C(R)^k K^k. \end{aligned}$$

The last inequality is equation (20) with $K = 3^{\frac{1}{2}} C_\varepsilon C_1 |-1|^{-\tau}$, in the notation used there. We have also used $\|b^*(p) \varphi_0\|_p = 1$.

This completes the proof of Lemma 26.

We give now some final results which are immediate consequences of Lemma 26 and of some theorems in chapter IV of [1]. (See also Kato [16, chapter VII]).

LEMMA 28. Let $|p| \leq R$, $|z - \omega(p)| = \varepsilon$. Then

$$(R_\lambda(z)_{\text{ren}} b^*(p))_p$$

is analytic in $\{|\lambda| \leq \lambda_0\}$.

Proof: This follows from the convergence of the Born series, because analyticity of bounded operator valued functions is preserved under composition and uniform limits.

THEOREM 29. We define the one-nucleon state ϕ_λ by

$$(\phi_\lambda)_p = (2\pi i)^{-1} \int_{|z-\omega(p)|=\epsilon} R_\lambda(z) b^*(p) \psi_0 dz .$$

$(\phi_\lambda)_p$ is analytic in $\{ |\lambda| < \lambda_0 \}$. The eigenvalue $E_{\lambda,p}$ of $(H_\lambda)_p$ corresponding to $(\phi_\lambda)_p$ is holomorphic in $\{ |\lambda| < \lambda_0 \}$.

Proof: [1], Theorem (4-1D) and (4-1E) .

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